Noname manuscript No.
(will be inserted by the editor)

Reduction of nonautonomous population dynamics models with two time scales

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Received: date / Accepted: date

Abstract The purpose of this work is reviewing some reduction results to deal with systems of nonautonomous ordinary differential equations with two time scales. They could be included among the so-called approximate aggregation methods. The existence of different time scales in a system, together with some long-term features, are used to build up a simpler system governed by a lesser number of state variables. The asymptotic behavior of the latter system is then used to describe the asymptotic behaviour of the former one. The reduction results are stated in two particular but important cases: periodic systems and asymptotically autonomous systems.

The reduction results are illustrated with the help of simple spatial SIS epidemic models including either periodic or asymptotically autonomous terms.

Keywords Slow-fast dynamics · Singular perturbations · Periodic systems · Asymptotically autonomous systems · Epidemic models

1 Introduction

The mathematical models used in population dynamics necessarily show the complexity found in natural systems. They are often governed by a large number of variables corresponding to different interacting organization levels. Some methods of reduction should be used in order to transform such models into mathematically tractable ones. The so-called aggregation of variables methods can be included in this latter category.

The term aggregation of variables appeared first in economy and later introduced in ecology [25, 20, 19]. The aggregation of a system consists of finding

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a certain number of global variables, functions of its state variables, and a system describing their dynamics. Aggregation is called perfect if the dynamics of the global variables is identical both in the initial system and in the aggregated one [20]. On the other hand, approximate aggregation [19] deals with methods of reduction where the consistency between the dynamics of the global variables in the initial and the aggregated systems is only approximate. In [3] it is suggested a whole program of study of aggregation methods linked to the existence of different time scales in the frame of autonomous ordinary differential equations. This study was motivated in the broad sense by the hierarchy theory in ecology. The method was rigourously justified in [7] in terms of an adequate version of the Fenichel centre manifold theorem [10, 11]. It allows to study the asymptotic behavior of the complete initial system with the help of a reduced system for some global variables called *aggregated system*. The approximate aggregation methods have also been developed for time discrete and infinite dimensional dynamical systems [4,5,36,6,8].

Autonomous systems represent population dynamics models where environment is taken to be constant. In order to consider environmental fluctuations the corresponding system must be nonautonomous which, in general, are much more difficult to analyze. Nevertheless, there are situations where it is possible to take advantage of some properties of the varying parameters, periodicity or being asymptotically constant, that simplify the analysis. The purpose of this work is reviewing and illustrating the aggregation techniques recently proposed [28,29,31,30] to deal with systems of nonautonomous ordinary differential equations with two time scales.

The utilized reduction techniques are mild versions of general results from singular perturbation methods dealing with slow-fast initial value problems of the form

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(t,x,y,\varepsilon), \;\; x(t_0) = x_0 \\ \dfrac{dy}{dt} = g(t,x,y,\varepsilon), \;\; y(t_0) = y_0 \end{cases}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and ε is a positive small parameter. Variable x exhibits a fast dynamics, represented by $f(t,x,y,0)$, while variable y moves at the slow time scale. In [40], for sufficiently smooth f and g and assuming the existence of a unique solution of the reduced problem (setting $\varepsilon = 0$)

$$\begin{cases} 0 = f(t,x,y,0), \\ \dfrac{dy}{dt} = g(t,x,y,0), \;\; y(t_0) = y_0 \end{cases}$$

the solutions of the initial system are uniformly approximated, when ε tends to 0, by those of the reduced system on bounded time intervals. A key assumption is that the solutions $x = \phi(t,y)$ of the equation $0 = f(t,x,y,0)$ are asymptotically stable equilibria of the equation $dx/d\tau = f(t,x,y,0)$ uniformly in t and y, considered here as parameters. The idea is that the fast dynamics acting on variable x has a negligible dependence on small variations of parameters t and y. This result is completed [42,43] with some others giving asymptotic expansions of the solutions including the fast behaviour of the boundary layer. The interest in applications of the analysis of the initial slow-fast system focus on its asymptotic behaviour. So, the reduction results just described need to be deepen so that they can be applied on infinite intervals of time. This is done for the first time in [15] with the help of a new stability hypothesis. The equation $dy/dt = g(t, \phi(t, y), y, 0)$ must possess a uniformly asymptotically stable solution. The demanding hypotheses needed to carry out the whole reduction process can be checked with simpler assumptions when dealing with particular forms of the general slow-fast system. This task is undertaken in [31, 30] and reviewed in this work in a simplified form.

In section 2 general results on approximate aggregation of nonautonomous systems of ordinary differential equations are presented. They are based upon the theory of singular perturbations and quasi-static state analysis as presented in [15–18]. The reduction results are particularized for periodic and asymptotically autonomous systems. A simple spatial epidemic model including periodic and asymptotically autonomous terms is presented in section 3. Its analysis is carried out with the help of the reduction results of the previous section.

2 System reduction

In this section we consider systems of differential equations of the form:

$$\varepsilon \frac{dn}{dt} = f(t, n) + \varepsilon s(t, n, \varepsilon), \tag{1}$$

where $n \in \mathbb{R}^m$ and $\varepsilon > 0$ is a small parameter. We can interpret that the dynamics is acting at two different time scales with variable t representing the slow time variable. Functions f and s are associated to the fast and slow dynamics, respectively.

The proposed reduction of system (1) follows the structure of the approximate aggregation methods established in the case of autonomous systems. See [4, 5] for reviews of the method with applications to different biological models. Though the reduction structure of both cases, autonomous and non autonomous, is the same the utilized mathematical techniques are different. For autonomous systems Fenichel centre manifold theorem [10, 11] allows to express the asymptotic behaviour of the solutions through the dynamics on a center manifold that admits a regular expansion in terms of the parameter ε [21, 42, 43]. In the case of nonautonomous systems we use the results on solutions of ordinary differential equations with small parameters on infinite intervals of time found in [15–18].

The first step in reducing system (1) is to transform it into the so-called slow-fast form. An appropriate change of variables $n \in \mathbb{R}^m \to (x, y) \in \mathbb{R}^{m-k} \times$

$\mathbb{R}^k$ yields the following system:

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = F(t,x,y) + \varepsilon R(t,x,y,\varepsilon), \\ \dfrac{dy}{dt} = S(t,x,y,\varepsilon), \end{cases} \tag{2}$$

where y represents slow variables in the following sense. Changing the time variable to the fast scale, $t = \varepsilon\tau$ and $d/d\tau = \varepsilon d/dt$, we obtain for y the following equation

$$\frac{dy}{d\tau} = \varepsilon \hat{S}(\tau,x,y,\varepsilon),$$

which tell us that considering fast dynamics as instantaneous with respect to slow dynamics, i.e., setting $\varepsilon = 0$, we get $dy/d\tau = 0$ and hence that y is constant at the fast time scale.

To find the appropriate transformation leading to the slow-fast form (2) of the system (1) could be, in general, difficult [33,34]. Nevertheless, in some applications, as it can be seen in the next section, the context leads to it straightforwardly. The search for the slow variables, those ones kept constant by the fast dynamics, yields the key of the transformation.

To describe the general reduction result stated in theorem 6 in the appendix we pose the following initial value problem:

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = F(t,x,y) + \varepsilon R(t,x,y,\varepsilon), & x(t_0) = x_0, \\ \dfrac{dy}{dt} = S(t,x,y,\varepsilon), & y(t_0) = y_0. \end{cases} \tag{3}$$

Formally setting $\varepsilon = 0$ in (3) we obtain a reduced problem or, as it is called in the theory of singular perturbations, a degenerate system [15,41]:

$$\begin{cases} 0 = F(t,x,y), \\ \dfrac{dy}{dt} = S(t,x,y,0), \qquad y(t_0) = y_0. \end{cases} \tag{4}$$

We should be able to solve the equation $0 = F(t,x,y)$ for x in terms of t and y. Hypothesis H2 of theorem 6 assumes the existence of a function $x = \Phi(t,y) \in \mathcal{C}^2$ solving this equation. It represents the fast equilibria for constant values of time t and of the slow variables y. Substituting those values of x into the initial value problem for y in (4) leads us to the reduced system:

$$\frac{dy}{dt} = S(t,\Phi(t,y),y,0), \qquad y(t_0) = y_0. \tag{5}$$

Under certain hypotheses, which are stated in detail in theorem 6, it is possible to approximate the solution $(x(t),y(t))$ of (3) by the solution $y(t)$ of (5) together with the corresponding fast equilibria $x(t) = \Phi(t,y(t))$.

The are two main conditions to be met. The first one requires a stability condition for system $dx/d\tau = F(t,x,y)$ when taking t and y as parameters: fast equilibria $x = \Phi(t, y)$ must be asymptotically stable uniformly in $t \in [t_0, T]$ and y in a certain compact set. This hypothesis allows the aforementioned approximation of solutions on bounded time intervals [40–42, 18].

The extension of the results to unbounded time intervals needs the previous condition to be true for $t \in [t_0, \infty)$ and a stability condition for system $dy/dt = S(t, \Phi(t, y), y, 0)$ [15–18]: it must possess an uniformly asymptotically stable solution for $t \in [t_0, \infty)$ with y_0 in its domain of attraction.

These two conditions are far from being easily met. In the following we state the general approximation result found in theorem 6 for two particular but useful settings. The reward of this particularization is getting approximation theorems which hypotheses are much more easily checked.

2.1 Periodic case

Here we consider systems encompassing environmental fluctuations of periodic type. This kind of fluctuation pervades all natural systems. To simplify as much as possible the hypotheses we assume that both F and S, the functions describing the fast and the slow dynamics in system (2), are periodic of the same period ω.

We use the following notation: $K_R = \{x \in \mathbb{R}_+^{m-k} : |x| \leq R\}$ and $K_{R'} = \{y \in \mathbb{R}_+^k : |y| \leq R'\}$, $K = K_R \times K_{R'}$, $\Omega = [t_0, \infty) \times K$, $\hat{K} = K \times [0, \varepsilon_0]$ and $\hat{\Omega} = \Omega \times [0, \varepsilon_0]$.

Theorem 1 *Let us consider system (3) where functions F and S are* periodic of the same period ω. *Let us assume:*

1. *$F \in \mathcal{C}^2(\Omega)$, R, $S \in \mathcal{C}^2(\hat{\Omega})$, and any solution of system (3) beginning in $K_R \times K_{R'}$ remains there for $t \in [t_0, \infty)$.*
2. *There is a function $x = \Phi(t, y) \in \mathcal{C}^2([t_0, \infty) \times K_{R'})$ such that for any $(t, y) \in [t_0, \infty) \times K_{R'}$ the following hold:*
   (a) *$F(t, \Phi(t, y), y) = 0$.*
   (b) *The real part of the eigenvalues of $J_x F(t, \Phi(t, y), y)$ is negative.*
3. *The system of equations*

$$\frac{d\bar{y}}{dt} = S(t, \Phi(t, \bar{y}), \bar{y}, 0), \tag{6}$$

has an asymptotically stable periodic solution $y^*(t)$ of period ω.

Let $(x^\varepsilon(t), y^\varepsilon(t))$ be the solution of system (3) for $(x^\varepsilon(t_0), y^\varepsilon(t_0)) = (x_0, y_0)$, with x_0 and y_0 in the domains of attraction, respectively, of the equilibrium $\Phi(t_0, y_0)$ of system $dx/d\tau = F(t_0, x, y_0)$ and of $y^*(t)$. Then, for any $\delta > 0$, there exist $\varepsilon_\delta > 0$ and $t_\delta > t_0$ such that

$$|(x^\varepsilon(t), y^\varepsilon(t)) - (\Phi(t, y^*(t)), y^*(t))| < \delta,$$

for every $\varepsilon \leq \varepsilon_\delta$ and every $t \geq t_\delta$.

On the one hand, the general hypothesis H3 of the theorem 6 is reduced to checking the sign of the real parts of the eigenvalues of the Jacobian matrix $J_xF(t,\Phi(t,y),y)$ and, on the other hand, the asymptotic stability of $y^*(t)$ implies, being ω-periodic, that it is uniform as required in hypothesis H4. The details of the proof of the theorem together with an application to a periodic prey-predator model with refuge can be found in [31].

2.2 Asymptotically autonomous case

In this section we consider the case of asymptotically autonomous (in the sequel A.A.) systems, which could be roughly described as nonautonomous systems such that their time varying terms tend to be constant [39,23], i.e., their time dependence disappears in the long term.

A continuous function $A:(t_0,\infty)\times D\to D$ with $(t_0,\infty)\times D\subset\mathbb{R}\times\mathbb{R}^N$, is said to be *asymptotically autonomous* if there exists a continuous function $\bar{A}:D\to D$ such that $\lim_{t\to\infty}A(t,z)=\bar{A}(z)$ uniformly on compact sets of D. If the function A is asymptotically autonomous then the nonautonomous system $x'=A(t,x)$ is also called *asymptotically autonomous*, being the autonomous system $x'=\bar{A}(x)$ its associated *limit system*. The main results on A.A. systems can be found in [27,32]. Under certain conditions the asymptotic behavior of an A.A. system can be obtained from its associated limit system.

To state a result similar to theorem 1 for A.A. systems let us start with a system in slow-fast form as (2) but expressed in terms of $\tau=t/\varepsilon$ the fast time variable

$$\begin{cases}\dfrac{dx}{d\tau}=F(\tau,x,y)+\varepsilon R(\tau,x,y,\varepsilon),\\ \dfrac{dy}{d\tau}=\varepsilon S(\tau,x,y,\varepsilon).\end{cases}\tag{7}$$

Theorem 2 *Let us consider system (7) where functions F and S are* asymptotically autonomous *on K and $\hat{K}$, respectively, being $\bar{F}(x,y)$ and $\bar{S}(x,y,\varepsilon)$ their corresponding asymptotic limit functions. Let us assume:*

1. *$F\in\mathcal{C}^2(\Omega)$, R, $S\in\mathcal{C}^2(\hat{\Omega})$, and any solution of system (7) beginning in $K_R\times K_{R'}$ remains there for forward time.*
2. *There is a function $x=\Phi(y)\in\mathcal{C}^2(K_{R'})$ such that for any $y\in K_{R'}$ the following hold:*
   (a) $\bar{F}(\Phi(y),y)=0$.
   (b) The real part of the eigenvalues of $J_x\bar{F}(\Phi(y),y)$ is negative.
3. *The system of equations*

$$\frac{d\bar{y}}{dt}=\bar{S}(\Phi(\bar{y}),\bar{y},0),\tag{8}$$

has an asymptotically stable solution $y^*(t)$.

Let $(x^\varepsilon(t), y^\varepsilon(t))$*,* $t = \varepsilon\tau$*, be the solution of system (7) for* $(x^\varepsilon(t_0), y^\varepsilon(t_0)) = (x_0, y_0)$*, with* x_0 *and* y_0 *in the domains of attraction, respectively, of the equilibrium* $\Phi(y_0)$ *of system* $d\bar{x}/dt = \bar{F}(\bar{x}, y_0)$ *and of* $y^*(t)$*. Then, for any* $\delta > 0$*, there exist* $\varepsilon_\delta > 0$ *and* $t_\delta > t_0$ *such that*

$$|(x^\varepsilon(t), y^\varepsilon(t)) - (\Phi(y^*(t)), y^*(t))| < \delta,$$

for every $\varepsilon \leq \varepsilon_\delta$ *and every* $t \geq t_\delta$*.*

The main conditions of theorem 2 are expressed in terms of autonomous systems related to the asymptotic limits of F and S. The details of the proof of the theorem together with applications to gradostat models can be found in [30].

3 Application to a simple epidemic model

In this section we illustrate the theorems stated in section 2 applying them to a simple epidemic model. We consider, on the one hand, the population split between susceptible and infective individuals and, on the other hand, that susceptible individuals can move between two different patches whereas infective individuals rest confined in one of them. Movements of susceptible individuals are assumed to be fast compared to the disease dynamics.

We call S_1 and S_2 the number of susceptible individuals in patches 1 and 2, respectively, and I the number of infected individuals, that stay in patch 2.

We consider in patch 2 a classical SIS model [9]. The susceptible-infective-susceptible (SIS) epidemiological model is the simplest description of the dynamics of a disease that is contact-transmitted and that confers no immunity against reinfection. It is appropriate for most diseases transmitted by bacterial or helminth agents, and most sexually transmitted diseases as gonorrhea, but not for diseases as AIDS for which there is no recovery.

The total population is assumed constant. We denote γ the recovery rate and $B(S_2, I)$ the incidence rate, that will be specified later on. Susceptible individuals leave patch 1 and 2 at rates m_1 and m_2, respectively. The ratio between time scales at which movements and disease dynamics act is represented by parameter $\varepsilon > 0$. Thus, the initial complete model reads as follows:

$$\begin{cases} \varepsilon \dfrac{dS_1}{dt} = -m_1 S_1 + m_2 S_2, \\ \varepsilon \dfrac{dS_2}{dt} = m_1 S_1 - m_2 S_2 + \varepsilon \left(-B(S_2, I) + \gamma I\right), \\ \dfrac{dI}{dt} = B(S_2, I) - \gamma I. \end{cases} \tag{9}$$

We could think of patch 1 as a sort of refuge for susceptible individuals, or else of patch 2 as a quarantine region. The model should serve to study the effect that susceptible individuals displacements have on the outcome of epidemics. All the coefficients of the model are assumed to depend on time.

We present first a periodic dependence and then we treat an asymptotically autonomous case.

We can find in the literature some other models concerning the study of two time scales spatially distributed epidemics models in which the individuals movements are considered fast. In [22] an autonomous model is considered, while periodic nonautonomous models are addressed in [28,29]. The novelty of the simple model (9) could be found in the quarantine zone. There also exists a vast literature encompassing studies of patchy distributed epidemic, mostly autonomous, models where displacements and epidemics act at the same time scale, see the recent review [2] and the references therein.

To transform system (9) into the slow-fast form described in (2) we make appear the slow variable $S = S_1 + S_2$, the total number of susceptible individuals, which is a constant of motion for the fast dynamics. Substituting S_2 by S in (9) we obtain:

$$\begin{cases} \varepsilon \dfrac{dS_1}{dt} = -m_1 S_1 + m_2(S - S_1), \\ \dfrac{dS}{dt} = -B(S - S_1, I) + \gamma I, \\ \dfrac{dI}{dt} = B(S - S_1, I) - \gamma I. \end{cases} \tag{10}$$

We notice that the total population $N = S + I$ keeps constant. Thus, the slow part of the system can be described in terms of the variable I and the constant N. Introducing this reduction the system (10) takes the simpler form:

$$\begin{cases} \varepsilon \dfrac{dS_1}{dt} = -m_1 S_1 + m_2(N - I - S_1), \\ \dfrac{dI}{dt} = B(N - I - S_1, I) - \gamma I. \end{cases} \tag{11}$$

3.1 Case of periodic rates.

The periodicity in the disease incidence rates is an issue broadly treated in the literature (see [14] for a review). The incidence of many infectious diseases often exhibits periodic patterns. Influenza is one of the diseases that it is well reported to have a seasonal pattern every year [24]. The number of measles cases per week also oscillates with a period of about two years [1]. It is also known that other childhood diseases such as mumps, chicken-pox, rubella, and pertussis exhibit seasonal behaviour. The periodicity of 1 year, which is fairly general for these diseases, might have its origin in the children contact rates depending on the duration of academic school years. Meningococcal meningitis in western Africa also varies seasonally [35,37] mainly due to atmospheric circulation. The yearly cyclic pattern of human immune system also yields seasonal oscillations of certain diseases [12,24].

The original SIS epidemic model assume the recovery rate γ to be constant, and the incidence rate to be of mass action type $B(S,I)=\beta SI$ with the parameter β, called transmission rate, also constant. One of its first modifications considered a periodic transmission rate $\beta(t)$ [13].

Here we consider the system (11) with incidence function $B(S,I)=\beta(t)SI$. All rates involved in the system are positive periodic functions of t sharing the same period ω. The nonautonomous periodic system reads as follows:

$$\begin{cases} \varepsilon \dfrac{dS_1}{dt} = -m_1(t)S_1 + m_2(t)(N-I-S_1), \\ \dfrac{dI}{dt} = \beta(t)(N-I-S_1)I - \gamma(t)I. \end{cases} \tag{12}$$

We follow the procedure described in section 2 and then apply theorem 1 to analyzed the asymptotic behaviour of the solutions of the system (12).

With the notations of section 2, we have

$$F(t,S_1,I) = -m_1(t)S_1 + m_2(t)(N-I-S_1),$$
$$S(t,S_1,I) = \beta(t)(N-I-S_1)I - \gamma(t)I.$$

The fast equilibria $S_1=\Phi(t,I)$, such that $F(t,\Phi(I),I)=0$, are

$$\Phi(t,I) = \frac{m_2(t)}{m_1(t)+m_2(t)}(N-I) = \mu(t)(N-I), \tag{13}$$

where we call $\mu(t)=m_2(t)/(m_1(t)+m_2(t))$, which represents the periodic long-term proportion of susceptible individuals in patch 1.

Taking t and I as constant and calculating the derivative of F with respect to S_1 we obtain

$$J_{S_1}F(t,\Phi(I),I) = -m_1(t) - m_2(t) < 0 \text{ for any } t \text{ and } I.$$

This last condition is needed to ensure that hypothesis 2 of theorem 1 is met. It is also necessary, in the last part of theorem, to know the domain of attraction of the fast equilibria. Having in mind that the equation

$$\frac{dS_1}{d\tau} = -m_1(t_0)S_1 + m_2(t_0)(N-I_0-S_1),$$

for constant t_0 and I_0, is linear in S_1, it is straightforward that the required domain of attraction is $(-\infty,\infty)$.

The next step, to deal with hypothesis 3 of theorem 1, is to construct the reduced system (6). For that we substitute S_1 in the second equation of the system (12) by the corresponding fast equilibria:

$$\frac{dI}{dt} = \beta(t)\left(N-I-\mu(t)(N-I)\right)I - \gamma(t)I,$$

that can also be expressed in the following form

$$\frac{dI}{dt} = \left(\beta(t)(1-\mu(t))N - \gamma(t)\right)I - \beta(t)(1-\mu(t))I^2. \tag{14}$$

This equation is known as the periodic Bernoulli equation and has received much attention in the literature [38,26]. To apply theorem 1 we need to find asymptotically stable periodic solutions of equation (14). For that, we follow [26] and define

$$< \beta(1-\mu) >= \frac{1}{\omega}\int_0^\omega \beta(t)(1-\mu(t))dt \quad \text{and} \quad < \gamma >= \frac{1}{\omega}\int_0^\omega \gamma(t)dt,$$

and from them the next dimensionless quantity that plays de role of basic reproductive number R_0 for the disease in system (12):

$$R_0 = \frac{< \beta(1-\mu) >}{< \gamma >} N. \tag{15}$$

We then have that $R_0 < 1$ implies ([26], Prop. 2.2) that the disease-free equilibrium $I^* = 0$ is globally asymptotically stable and if $R_0 > 1$ then ([26], Th. 3.1) there exists a unique positive periodic function $I_p(t)$ which attracts every solution of equation (14) with a positive initial value.

Applying theorem 1 we obtain the asymptotic behaviour of solutions of the system (12) in terms of R_0.

Theorem 3 *Let* $(S_1^\varepsilon(t), I^\varepsilon(t))$ *be the solution of system* (12) *with initial values* $S_1(t_0) \geq 0$ *and* $I(t_0) > 0$. *Then, for any* $\delta > 0$, *there exist* $\varepsilon_\delta > 0$ *and* $t_\delta > t_0$ *such that, for every* $\varepsilon \leq \varepsilon_\delta$ *and every* $t \geq t_\delta$,

1. *If* $R_0 < 1$
$$|\,(S_1^\varepsilon(t), I^\varepsilon(t)) - (\mu(t)N, 0)\,| < \delta.$$
2. *If* $R_0 > 1$
$$|\,(S_1^\varepsilon(t), I^\varepsilon(t)) - (\mu(t)(N - I_p(t)), I_p(t))\,| < \delta.$$

R_0 incorporates the information of the fast dynamics, displacements between patches, through the fraction of susceptible individuals $1 - \mu(t)$ staying in patch 2. Thus, theorem 3 summarizes the influence of migration rates on the outcome of the epidemic process, what could be useful for managing decisions as, for instance, quarantine measures.

In figure 1 there is a numerical example comparing the solution of the complete system (12) with the approximation obtained through the reduced system (14) with a time scale ratio $\varepsilon = 0.1$.

3.2 Case of asymptotically autonomous rates.

In this section we consider the system (11) expressed in terms of the fast time variable. We analyze two different cases, the first one with mass action incidence function and the second one with proportional incidence. In both cases we assume positive asymptotically autonomous rates.

- *Mass action incidence function*

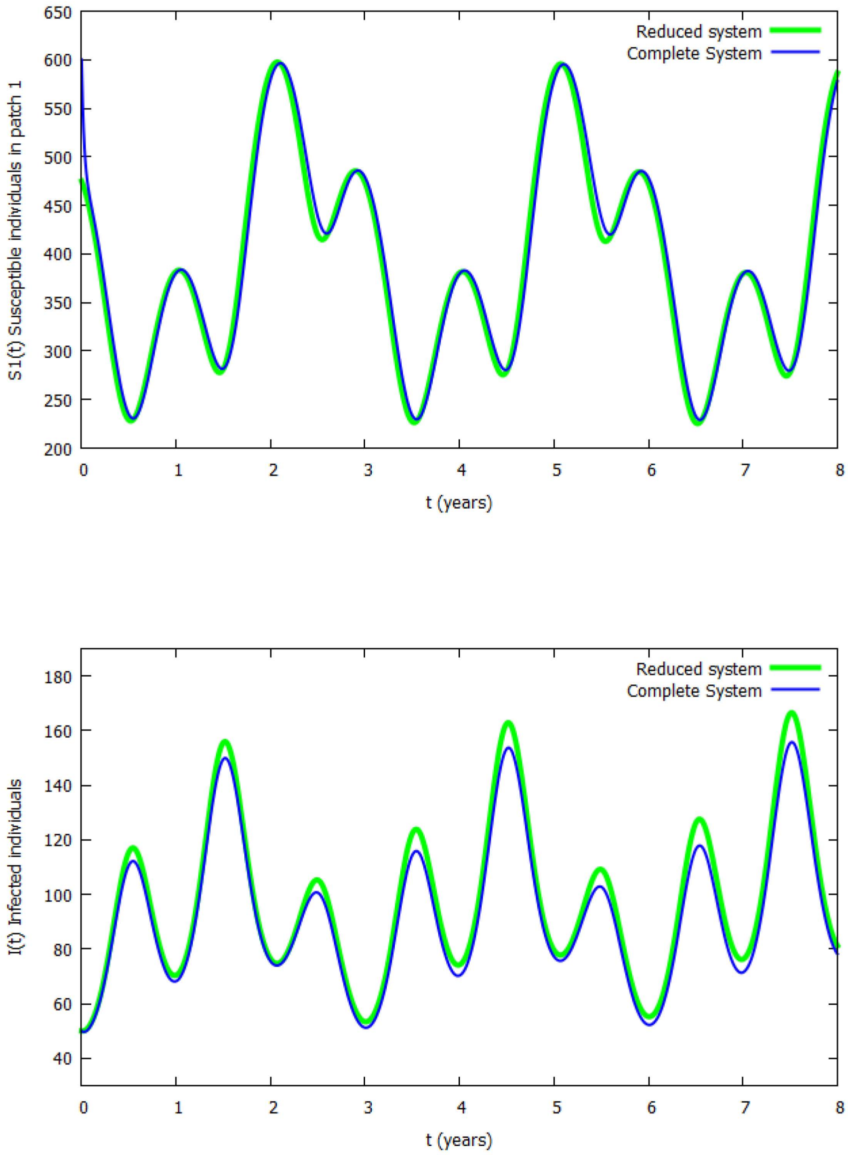

Fig. 1 Comparison of the solution $(S_1(t), I(t))$ of the complete system (12), $\varepsilon = 0.1$ and $(S_1(0), I(0)) = (600, 50)$, with $(\mu(t)(N - I(t)), I(t))$ where $I(t)$ is the solution of the reduced system (14), $I(0) = 50$. The total population: $N = 1000$. The migrations rates: $m_1(t) = 2+\sin(\frac{2}{3}\pi t)$ and $m_2(t) = \frac{3}{2} + \frac{1}{2}\cos(2\pi t)$. The transmission rate: $\beta(t) = 0.004(1+\frac{1}{2}\sin(2\pi t))$. The recovery rate: $\gamma(t) = 2(1 - \frac{1}{2}\sin(2\pi t))$. The basic reproductive number (15) is $R_0 = 1.2263611$.

The corresponding system is:

$$\begin{cases} \dfrac{dS_1}{d\tau} = -m_1(\tau)S_1 + m_2(\tau)(N - I - S_1), \\ \dfrac{dI}{d\tau} = \varepsilon\left(\beta(\tau)(N - I - S_1)I - \gamma(\tau)I\right), \end{cases} \tag{16}$$

where all rates are supposed to have limit when $\tau \to \infty$, hence yielding an asymptotically autonomous system of the form (7). For further purposes, we note

$$\bar{z} := \lim_{\tau\to\infty} z(\tau), \qquad z(\tau) \in \{m_1(\tau), m_2(\tau), \beta(\tau), \gamma(\tau)\}. \tag{17}$$

The asymptotic stability of the different rates can be interpreted as the fact that health authorities can modify displacement rates rapidly as well as salubrity conditions by implementing prophylaxis procedures in a rather short period of time.

We now apply theorem 2 to study the system (16).

We have

$$\bar{F}(S_1, I) = -\bar{m}_1 S_1 + \bar{m}_2(N - I - S_1) \text{ and } \bar{S}(S_1, I) = \bar{\beta}(N - I - S_1)I - \bar{\gamma}I.$$

The fast equilibria $S_1 = \Phi(I)$ are

$$\Phi(I) = \frac{\bar{m}_2}{\bar{m}_1 + \bar{m}_2}(N - I) = \mu(N - I), \tag{18}$$

where $\mu = \bar{m}_2/(\bar{m}_1+\bar{m}_2)$ is the long-term proportion of susceptible individuals in patch 1.

In this case $J_{S_1}\bar{F}(\Phi(I), I) = -\bar{m}_1 - \bar{m}_2 < 0$ and the domain of attraction of the equilibrium $\Phi(I)$ in equation $dS_1/d\tau = -m_1(\tau)S_1 + m_2(\tau)(N - I - S_1)$ is $(-\infty, \infty)$.

The reduced system (8) for system (16) is

$$\frac{dI}{dt} = \bar{\beta}(N - I - \mu(N - I))I - \bar{\gamma}I,$$

that can be written in logistic form

$$\frac{dI}{dt} = \left(\bar{\beta}(1 - \mu)N - \bar{\gamma}\right) I \left(1 - \frac{I}{N - \bar{\gamma}/(\bar{\beta}(1 - \mu))}\right). \tag{19}$$

This equation possesses an asymptotically stable equilibrium I^* which domain of attraction includes every positive initial value. The equilibrium I^* can be either 0 or $N - \bar{\gamma}/(\bar{\beta}(1-\mu))$ depending on the latter being negative or positive. Equivalently, this condition can be expressed in terms of the next dimensionless quantity that could be taken as the basic reproductive number R_0 for the disease in system (16):

$$R_0 = \frac{\bar{\beta}(1 - \mu)}{\bar{\gamma}}N. \tag{20}$$

If $R_0 < 1$ the infection dies out and thus $I^* = 0$, the disease-free equilibrium. On the other hand, if $R_0 > 1$ the infection persists and $I^* = N - \bar{\gamma}/(\bar{\beta}(1-\mu))$ is then the so-called endemic equilibrium.

We can now conclude from theorem 2 the following analysis of the asymptotic behaviour of solutions of system (16) in terms of R_0.

Theorem 4 *Let $(S_1^{\varepsilon}(t), I^{\varepsilon}(t))$, $t = \varepsilon\tau$, be the solution of system* (16) *with initial values $S_1(t_0) \geq 0$ and $I(t_0) > 0$. Then, for any $\delta > 0$, there exist $\varepsilon_\delta > 0$ and $t_\delta > t_0$ such that*

$$|\,(S_1^{\varepsilon}(t), I^{\varepsilon}(t)) - (\mu(N - I^*), I^*)\,| < \delta,$$

for every $\varepsilon \leq \varepsilon_\delta$ and every $t \geq t_\delta$, where:

1. $I^* = 0$ *if* $R_0 < 1$.
2. $I^* = N\left(1 - \dfrac{1}{R_0}\right)$ *if* $R_0 > 1$.

As in the periodic theorem 4 summarizes the influence of migration rates on the outcome of the epidemic process.

In figure 2 there is a numerical example showing that the solution of the complete system (16) approaches the equilibrium point predicted by the reduced system (19).

- *Proportional incidence*

In this case the corresponding system is:

$$\begin{cases} \dfrac{dS_1}{d\tau} = -m_1(\tau)S_1 + m_2(\tau)(N - I - S_1), \\ \dfrac{dI}{d\tau} = \varepsilon\left(\beta(\tau)\dfrac{(N - I - S_1)I}{N - S_1} - \gamma(\tau)I\right). \end{cases} \tag{21}$$

We keep the notation (17). In order to apply theorem 2 to study the system (21) we notice that its fast part coincides with that of system (16). Thus, we have the same fast equilibria $S_1 = \Phi(I) = (\bar{m}_2/(\bar{m}_1 + \bar{m}_2))(N - I) = \mu(N - I)$, which are globally asymptotically stable.

The corresponding reduced system (8) for system (21) is

$$\frac{dI}{dt} = \bar{\beta}\frac{(N - I - \mu(N - I))I}{N - \mu(N - I)} - \bar{\gamma}I,$$

that can also be written as

$$\frac{dI}{dt} = \left(\frac{\bar{\beta}(1-\mu)(N - I)}{(1-\mu)N + \mu I} - \bar{\gamma}\right)I. \tag{22}$$

This equation also possesses two equilibria:

$$I_1^* = 0 \ \text{ and } \ I_2^* = \frac{(\bar{\beta} - \bar{\gamma})(1-\mu)}{\bar{\beta}(1-\mu) + \bar{\gamma}\mu}N.$$

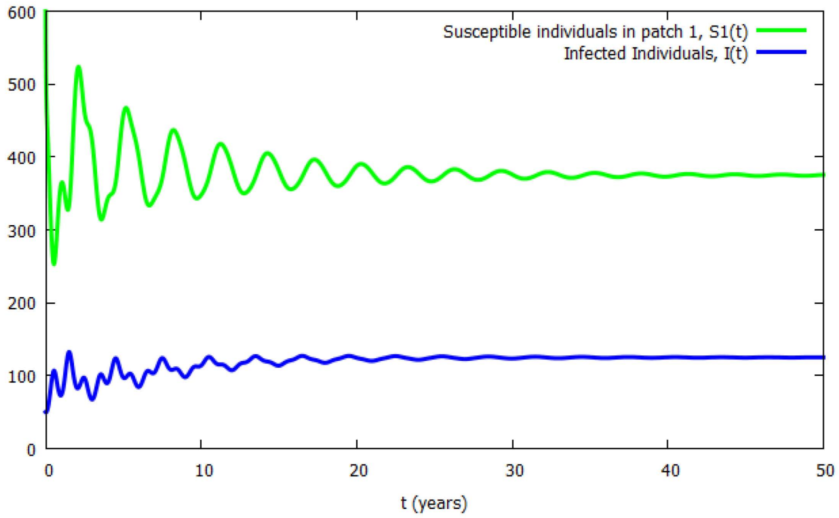

Fig. 2 Solution $(S_1(t), I(t))$ of the complete system (16), $\varepsilon = 0.1$ and $(S_1(0), I(0)) = (600, 50)$. The total population: $N = 1000$. The migrations rates: $m_1(t) = 2+e^{-0.1t}\sin(\frac{2}{3}\pi t)$ and $m_2(t) = \frac{3}{2}+\frac{1}{2}e^{-0.5t}\cos(2\pi t)$. The transmission rate: $\beta(t) = 0.004(1+\frac{1}{2}e^{-0.3t}\sin(2\pi t))$. The recovery rate: $\gamma(t) = 2(1-\frac{1}{2}e^{-0.2t}\sin(2\pi t))$. The basic reproductive number (20) is $R_0 = \frac{8}{7}$. The asymptotic behaviour of the solution is well described by the equilibrium obtained through the reduced system (19): $I^* = 125$ and $S_1^* = \mu(N - I^*) = 375$.

The disease-free equilibrium $I_1^* = 0$ is asymptotically stable whenever $\bar{\beta} < \bar{\gamma}$. In the opposite case, $\bar{\beta} > \bar{\gamma}$, the equilibrium I_2^* is positive and asymptotically stable and thus it becomes the endemic equilibrium.

The condition to distinguish between disease eradication or endemicity can be expressed in terms of the following basic reproductive number R_0 for system (21):

$$R_0 = \frac{\bar{\beta}}{\bar{\gamma}}. \tag{23}$$

The asymptotic behaviour of solutions of system (21) are summarized in the next theorem, which is analogous to theorem 4.

Theorem 5 *Let* $(S_1^\varepsilon(t), I^\varepsilon(t))$, $t = \varepsilon\tau$, *be the solution of system* (21) *with initial values* $S_1(t_0) \geq 0$ *and* $I(t_0) > 0$. *Then, for any* $\delta > 0$, *there exist* $\varepsilon_\delta > 0$ *and* $t_\delta > t_0$ *such that*

$$|\,(S_1^\varepsilon(t), I^\varepsilon(t)) - (\mu(N - I^*), I^*)\,| < \delta,$$

for every $\varepsilon \leq \varepsilon_\delta$ *and every* $t \geq t_\delta$, *where:*

1. $I^* = 0$ *if* $R_0 < 1$.

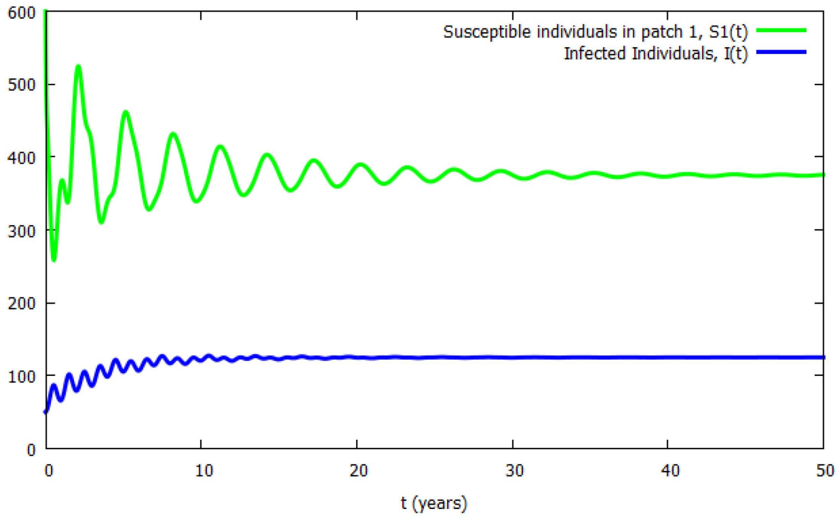

Fig. 3 Solution $(S_1(t), I(t))$ of the complete system (21), $\varepsilon = 0.1$ and $(S_1(0), I(0)) = (600, 50)$. The total population: $N = 1000$. The migrations rates: $m_1(t) = 2 + e^{-0.1t}\sin(\frac{2}{3}\pi t)$ and $m_2(t) = \frac{3}{2} + \frac{1}{2}e^{-0.5t}\cos(2\pi t)$. The transmission rate: $\beta(t) = \frac{5}{2} + \frac{1}{2}e^{-0.3t}\sin(2\pi t))$. The recovery rate: $\gamma(t) = 2(1 - \frac{1}{2}e^{-0.2t}\sin(2\pi t))$. The basic reproductive number (23) is $R_0 = 1.2$. The asymptotic behaviour of the solution is well described by the equilibrium obtained through the reduced system (22): $I^* = 125$ and $S_1^* = \mu(N - I^*) = 375$.

2. $I^* = N\left(1 - \dfrac{1}{(1-\mu)R_0 + \mu}\right)$ *if* $R_0 > 1$.

We notice that, in this case, displacements of susceptible individuals play no role in the long-term behaviour of the disease. It is the ratio of transmission rate to recovery rate in patch 2 which decides between disease eradication and endemicity. Proportional incidence is adequate for large populations or low incidence rates. In those cases the existence of two patches joined by fast displacements has no crucial effect on the disease evolution. Its influence appears just in the number of infective individuals. Interpreting patch 2 as a quarantine area, managing efforts should be put in reducing the transmission or the mean recovery time in this area. Acting on the numbers of individuals out of the quarantine area would be useless.

In figure 3 there is a numerical example showing that the solution of the complete system (21) approaches the equilibrium point predicted by the reduced system (22).

4 Discussion and Conclusions

In this work it is reviewed how to apply the results on singular perturbations on unbounded intervals, developed in [15], to some slow-fast population dynamics models. The first step in the application of these results is setting the system of ordinary differential equations into the slow-fast form, that is, making emerge the slow variables, those not affected by the fast the dynamics. This allows, under certain hypotheses, to carry out the analysis of the asymptotic behavior of the solutions of the system in two steps corresponding roughly to its fast and slow parts. Concerning the first one, the asymptotically stable equilibria of the fast part of the system are found in terms of time t and the slow variables. For the slow part a reduced system is obtained by substituting the non slow variables by their corresponding equilibria. The asymptotic behaviour of this reduced system together with the fast equilibria allows to describe the asymptotic behavior of the initial system.

The presented reduction procedure justify on theoretical grounds what is a frequent practice: decoupling processes that act at different time scales though they are certainly coupled. The precise assumptions to be met, in order to ensure that the whole procedure is justified, are collected in theorem 6 following [18] and adapted in theorems 1 and 2 to two special situations in population dynamics: environments changing periodically and environments tending to stabilization. Hypotheses 3 and 4 of theorem 6 are difficult to check because the required asymptotic stabilities must be uniform. In theorems 1 and 2 the condition to met hypothesis 3 is just expressed in terms of the sign of the real parts of the eigenvalues of a Jacobian matrix, the uniformity being always met as well as in the asymptotic stability of the solution of the reduced system involved in hypothesis 4.

Similar results to theorems 1 and 2 have been already used in aplications. In [31] a predator-prey model with a prey refuge and slowly varying periodic coefficients is analyzed essentially following the reduction procedure summarize in theorem 1. Two other works, [28, 29], considering spatially distributed epidemic models with periodic coefficients are treated in an analogous form. On the other hand, a couple of gradostat models with asymptotically autonomous coefficients are analyzed in [30] applying the reduction technique described in theorem 2.

In this work, to illustrate the proposed reduction procedure by applying both theorems 1 and 2 we have chosen a simple application: a SIS epidemic model together with fast migrations between two different patches. It is about the simplest situation that contains all the required characteristics and still might have an interest as application in itself. Based upon the same model three cases are treated. In the first one, with mass action incidence function, all rates are considered periodic whereas in the other two are assumed asymptotically autonomous, one also with mass action incidence function and the other with proportional incidence.

In all the three cases the reduction procedure yields a basic reproductive numbers R_0 that ensures the disease eradication when being less than one and its endemicity if larger. In the periodic case the endemicity is determined by an asymptotically stable periodic solution of the reduced system. In both asymptotically autonomous cases the endemicity situation leads to a steady state. In these R_0 are summarized the join effects of the disease and the displacements of susceptible individuals between patches. From the point of view of disease management once the local epidemiological and demographic parameters are estimated, a control of epidemics can be considered by an adequately acting on individual displacements.

As mentioned in the introduction, the reduction procedure can be considered as an approximate aggregation method for nonautonomous ordinary differential equations. Having in mind the large number of interesting applications in the literature that are based upon the aggregation methods for autonomous differential equations [4–6] we do expect the procedure to be further developed and applied to more realistic models.

A Appendix

We summarize in the next theorem the results on singular perturbations methods for slow-fast dynamics on the infinite interval as presented for the first time in the work of Hoppensteadt [15], that the author subsequently included in a more readable way in reviews of differential equations with small parameters and quasi-static state analysis of differential equations [16–18]. In [43] it is found a review of singular perturbation methods for slow-fast systems where they are mentioned some other works, notably by Tikhonov [41], that preceded those of Hoppensteadt though for bounded intervals of time. It is also found in [43] the peculiarities of this theory applied to autonomous equations following the works by Fenichel [10].

Theorem 6 *Let us consider the initial-value problem*

$$\begin{cases} \varepsilon \dfrac{dx}{dt} = f(t,x,y,\varepsilon),\; x(t_0) = x_0, \\ \dfrac{dy}{dt} = g(t,x,y,\varepsilon),\; y(t_0) = y_0, \end{cases} \tag{24}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and ε is a small positive parameter. We call $\hat{\Omega} = \Omega \times [0, \varepsilon_0]$ where $\Omega = I \times B_R \times B_{R'}$, $I = [t_0, \infty\}$, $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$, $B_{R'} = \{y \in \mathbb{R}^m : |y| \le R'\}$ and ε_0 is a positive constant. Balls B_R and $B_{R'}$ could be replaced by any sets that are diffeomorphic to them.
Hypothesis H1. *f, $g \in \mathcal{C}^2(\hat{\Omega})$ and the solutions of the system (24) beginning in $B_R \times B_{R'}$ remains there for $t \in I$.*
Hypothesis H2. *There is a function $x = \Phi(t,y) \in \mathcal{C}^2$ such that $f(t, \Phi(t,y), y, 0) = 0$ for $(t,y) \in I \times B_{R'}$.*
Hypothesis H3. *$x = \Phi(t,y)$ is an asymptotically stable equilibrium of the system $\dfrac{dx}{d\tau} = f(t,x,y,0)$ uniformly in $(t,y) \in I \times B_{R'}$ and x_0 is in the domain of attraction of $\Phi(t_0, y_0)$.*
Hypothesis H4. *The system of equations $d\bar{y}/dt = g(t, \Phi(t,\bar{y}), \bar{y}, 0)$ has an uniformly asymptotically stable solution $y^*(t)$ for $t_0 \le t < \infty$ and y_0 is in its domain of attraction.*

Then if $\bar{y}(t)$ is the solution of

$$d\bar{y}/dt = g(t, \Phi(t,\bar{y}), \bar{y}, 0),\; \bar{y}(t_0) = y_0,$$

for sufficiently small values of ε the solution $(x(t), y(t))$ of the system (24) satisfies

$$x(t) = \Phi(t, \bar{y}(t)) + o(1), \qquad y(t) = \bar{y}(t) + o(1),$$

as $\varepsilon \to 0^+$ uniformly on any interval of the form $t_0 < t_1 \le t < \infty$.

Acknowledgements Authors are partially supported by Ministerio de Ciencia e Innovación (Spain), projects MTM2011-24321 and MTM2011-25238.

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