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Time scales in linear delayed differential equations ☆

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This paper is devoted to the loving memory of our friend Ovide Arino, who passed away during the production of it

Abstract

The aim of this paper is to apply and justify the so-called aggregation of variables method for reduction of a complex system of linear delayed differential equations with two time scales: slow and fast. The difference between these time scales makes a parameter $\varepsilon > 0$ to appear in the formulation, being a mathematical problem of singular perturbations. The main result of this work consists of demonstrating that, under some hypotheses, the solution to the perturbed problem converges when $\varepsilon \to 0$ to the solution of an aggregated system whose construction is proposed.

Keywords: Singular perturbations; Aggregation of variables; Delayed differential equations; Two time scales

1. Introduction

Nature offers many examples of systems where several events occur at different time scales. It is then common practice to consider those events occurring at the fastest scale as being instantaneous with respect to the slower ones, which results in a lesser number of variables or parameters

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needed to describe the evolution of the system. A subsequent issue is to determine how far the results obtained from the reduced system are from the real ones. Aggregation methods have been developed in relation with the above-mentioned issues, that is, reduction and estimation of the discrepancy between the complete system and the system arising from reduction. These methods have been investigated by some of us in the case of systems of ordinary differential equations with different time scales (see [4,6,7]), in the context of discrete dynamical systems [12–14,24] and for continuous time models of structured population dynamics, formulated as systems of partial differential equations [3,11]. A review of aggregation methods can be found in [5].

In this paper we extend the aggregation of variables method to the setting of linear delayed functional differential equations with two time scales. We remind the reader that delay differential equations are in some sense associated with the early development of mathematical ecology through the predator-prey models proposed in the twenties by V. Volterra [27]. The subject has grown since and it has been extended in various ways, for example, to partial differential equations with functional delay terms (see [15,16]), which justifies the need to introduce abstract formulations as presented in this work. A study of how delays emerge from inner ecological mechanisms was performed in the eighties by W. Gurney, R. Nisbet and coworkers [17,21]. In our model, the difference between time scales (slow and fast dynamics) makes a parameter $\varepsilon > 0$ to appear in the formulation, being a mathematical problem of singular perturbations.

The organization of the paper is as follows: Section 2 describes a first version with a discrete delay of the model to be considered, for which the aggregated system is constructed. In Section 3 the relationship between the solutions of this model and its aggregated system is analyzed. In Section 4 the aggregation of variables method is applied to an abstract linear delayed functional differential equation. Section 5 suggests an example of the application of the theoretical results of the previous sections to a model for the dynamics of a population structured in two stages, which inhabits an ecosystem divided into two different patches.

Although the model analyzed in Section 2 is a particular case of the general formulation developed in Section 4, we have considered it necessary to present both studies separately. On one side, the case of discrete delay contained in Section 2 is interesting in itself and clarifies the abstract formulation while on the other, it has its own methods for the step-by-step construction of the solution, which have been useful for us to proof Lemma 1, an essential piece in justifying the approximation results.

2. Description of the problem and construction of the aggregated model

We suppose that we are dealing with a hierarchically organized system in the context of natural processes whose dynamics could be described in a linear and time-continuous way. The model consists of the following system of linear delayed differential equations, depending on a small parameter $\varepsilon > 0$, that we call the *perturbed system*:

$$\begin{cases} X'(t) = (1/\varepsilon)KX(t) + AX(t) + BX(t-r), & t > 0, \\ X(t) = \Phi(t), \quad t \in [-r, 0], \ \Phi \in C([-r, 0]; \mathbf{R}^N), \end{cases} \tag{1}$$

where $X(t) := (\mathbf{x}_1(t), \ldots, \mathbf{x}_q(t))^T$, $\mathbf{x}_j(t) := (x_j^1(t), \ldots, x_j^{N_j}(t))^T$, $j = 1, \ldots, q$; K, A and B are $N \times N$ real constant matrices with $N = N_1 + \cdots + N_q$.

As usual, $C([-r, 0]; \mathbf{R}^N)$ represents the Banach space of $\mathbf{R}^N$-valued continuous functions on $[-r, 0]$, $(r > 0)$, endowed with the norm $\|\varphi\|_C := \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|$.

System (1) can be solved by the classical step-by-step procedure. We refer the reader to [10] for the general theory.

2.1. Structure of matrix K

Throughout this paper, we suppose that matrix K is a block-diagonal matrix

$$K := \begin{pmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & K_q \end{pmatrix}$$

in which each diagonal block K_j has dimensions $N_j \times N_j$, $j = 1, \dots, q$, and satisfies the following hypothesis.

Hypothesis 1. *For each $j = 1, \dots, q$, the following holds*:

(i) $\sigma(K_j) = \{0\} \cup \Lambda_j$, *with* $\Lambda_j \subset \{z \in \mathbf{C};\ \operatorname{Re} z < 0\}$, *where* $\sigma(K_j)$ *is the spectrum of matrix* K_j.
(ii) 0 *is a simple eigenvalue of* K_j.

As a consequence, $\ker K_j$ is generated by an eigenvector associated to eigenvalue 0, which will be denoted $\mathbf{v}_j$. The left eigenspace of matrix K_j associated to the eigenvalue 0 is generated by a vector $\mathbf{v}_j^*$, so that $K_j \mathbf{v}_j = \mathbf{0}_j$, $(\mathbf{v}_j^*)^T K_j = \mathbf{0}_j^T$, $\mathbf{0}_j := (0, \dots, 0)^T \in \mathbf{R}^{N_j}$.

We choose both vectors verifying the *normalization condition*: $(\mathbf{v}_j^*)^T \mathbf{v}_j = 1$.

Example. Hypothesis 1 holds for a matrix K if each diagonal block K_j is an irreducible matrix with non-negative elements outside the diagonal and in addition satisfies $\mathbf{1}_j^T K_j = \mathbf{0}_j^T$, $\mathbf{1}_j := (1, \dots, 1)^T \in \mathbf{R}^{N_j}$. In this case $\mathbf{v}_j^* = \mathbf{1}_j$.

In order to build the so-called *aggregated system* of system (1), we define the following matrices:

$$\mathcal{V}^* := \begin{pmatrix} (\mathbf{v}_1^*)^T & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & (\mathbf{v}_2^*)^T & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & (\mathbf{v}_q^*)^T \end{pmatrix}; \qquad \mathcal{V} := \begin{pmatrix} \mathbf{v}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{v}_q \end{pmatrix},$$

which satisfy $K\mathcal{V} = 0$, $\mathcal{V}^* K = 0$, $\mathcal{V}^*\mathcal{V} = I_q$ ($q \times q$ identity matrix).

As a consequence of Hypothesis 1, we can consider the following direct sum decomposition of space $\mathbf{R}^N$:

$$\mathbf{R}^N = \ker K \oplus S \tag{2}$$

where $\ker K$ is a q-dimensional subspace generated by the columns of matrix $\mathcal{V}$ and $S := \operatorname{Im} K = \{\mathbf{v} \in \mathbf{R}^N;\ \mathcal{V}^*\mathbf{v} = \mathbf{0}\}$.

Let us observe that K_S (restriction of K to S) is an isomorphism on S and also that there exists $\alpha > 0$ such that

$$\sigma(K_S) \subset \{z \in \mathbf{C};\ \operatorname{Re} z < -\alpha\}. \tag{3}$$

According to this decomposition, each vector $\mathbf{u} \in \mathbf{R}^N$ can be written as $\mathbf{u} = \mathcal{V}(a_1, \dots, a_q)^T + \mathbf{u}_S$, $\mathbf{u}_S \in S$, where the coefficients a_j, $j = 1, \dots, q$, satisfy $\mathcal{V}^*\mathbf{u} = \mathcal{V}^*\mathcal{V}(a_1, \dots, a_q)^T + \mathcal{V}^*\mathbf{u}_S = (a_1, \dots, a_q)^T$ and hence $\mathbf{u} = \mathcal{V}\mathcal{V}^*\mathbf{u} + \mathbf{u}_S$.

2.2. The aggregated model

We now define a new set of q variables, that we will call *aggregated variables*: $s_j(t) := (\mathbf{v}_j^*)^T \mathbf{x}_j(t)$, $j = 1, \ldots, q$, or in vector form:

$$\mathbf{s}(t) := \big(s_1(t), \ldots, s_q(t)\big)^T = \mathcal{V}^* X(t).$$

The linear differential system satisfied by these new variables is obtained by premultiplying both sides of (1) by $\mathcal{V}^*$:

$$\begin{aligned}\mathbf{s}'(t) &= \mathcal{V}^* X'(t) = \mathcal{V}^*\left(\frac{1}{\varepsilon} K X(t) + A X(t) + B X(t-r)\right)\\ &= \mathcal{V}^* A X(t) + \mathcal{V}^* B X(t-r).\end{aligned} \tag{4}$$

We get the aggregated variables on the left-hand side but we fail to on the right-hand side. To avoid this difficulty, we write $X(t)$ according to the decomposition (2):

$$X(t) = \mathcal{V}\mathcal{V}^* X(t) + X_S(t) = \mathcal{V}\mathbf{s}(t) + X_S(t)$$

so that

$$\mathbf{s}'(t) = \mathcal{V}^* A \mathcal{V}\mathbf{s}(t) + \mathcal{V}^* B \mathcal{V}\mathbf{s}(t-r) + \mathcal{V}^* A X_S(t) + \mathcal{V}^* B X_S(t-r).$$

Let us observe that (4) gives for $t \in [0, r]$,

$$\mathbf{s}'(t) = \mathcal{V}^* A \mathcal{V}\mathbf{s}(t) + \mathcal{V}^* A X_S(t) + \mathcal{V}^* B \Phi(t-r).$$

Therefore, we propose as *aggregated model* the following approximated system, with the aggregated variables as the unique state variables:

$$\mathbf{s}'(t) = \overline{A}\mathbf{s}(t) + \overline{B}\mathbf{s}(t-r), \quad t > r, \tag{5}$$

where $\overline{A} := \mathcal{V}^* A \mathcal{V}$, $\overline{B} := \mathcal{V}^* B \mathcal{V}$, and

$$\mathbf{s}'(t) = \overline{A}\mathbf{s}(t) + \mathcal{V}^* B \Phi(t-r), \quad t \in [0, r]. \tag{6}$$

Equation (5) is a delayed linear differential system of equations to which the general theory of [10] can also be applied. In particular, it can be solved by a step-by-step procedure from an initial data in $[0, r]$ which is the solution to (6), that is,

$$\mathbf{s}(t) = e^{t\overline{A}}\left[\mathcal{V}^*\Phi(0) + \int_0^t e^{-\sigma\overline{A}}\mathcal{V}^* B \Phi(\sigma - r)\, d\sigma\right].$$

3. Comparison between the solutions to systems (1) and (5)

The main goal of this paper is to obtain a comparison between the solutions to both systems (1) and (5). In this section we will obtain this result: for $t > 0$ and $\varepsilon > 0$ small enough, the solution $X_\varepsilon(t)$ to the perturbed system (1) can be decomposed into a stable part which is precisely $\mathcal{V}\mathbf{s}_0(t)$, $\mathbf{s}_0(t)$ being the solution to the aggregated model (5) and a perturbation of order $O(\varepsilon)$.

This approximation result is similar to that obtained in [3] for continuous time structured models, formulated in terms of partial differential equations, but we have to point out that the delay introduces significant differences due to the influence of the initial data on the solution in the interval $[0, r]$: the approximation when $\varepsilon \to 0$ is valid only for $t \geqslant r$ and hence the initial data in $[0, r]$ for the aggregated system is $\mathcal{V}^* X_\varepsilon(t)$, which is the projection on $\ker K$ of the exact solution to system (1), constructed in $[0, r]$ from an initial data $\Phi \in C([-r, 0]; \mathbf{R}^N)$.

First of all, we decompose the solution to (1) according to the direct sum decomposition (2), which is

$$X_\varepsilon(t) = \mathcal{V}\mathbf{s}_\varepsilon(t) + \mathbf{q}_\varepsilon(t), \quad t \geqslant 0, \tag{7}$$

where $\mathbf{s}_\varepsilon(t) = \mathcal{V}^* X_\varepsilon(t)$, $\mathcal{V}^*\mathbf{q}_\varepsilon(t) = \mathbf{0}$.

Notice that the last equality implies that $\mathcal{V}^*\mathbf{q}'_\varepsilon(t) = 0$, $t > 0$, which means that $\forall t > 0$, $\mathbf{q}'_\varepsilon(t) \in S$.

Substituting (7) into system (1), we get

$$X'_\varepsilon(t) = A\mathcal{V}\mathbf{s}_\varepsilon(t) + B\mathcal{V}\mathbf{s}_\varepsilon(t-r) + A\mathbf{q}_\varepsilon(t) + B\mathbf{q}_\varepsilon(t-r) + \frac{1}{\varepsilon}K\mathbf{q}_\varepsilon(t),$$

where we have taken into account that $(1/\varepsilon)K\mathcal{V}\mathbf{s}_\varepsilon(t) = \mathbf{0}$.

We project this equation onto subspaces $\ker K$ and S, by premultiplying successively by $\mathcal{V}^*$ and by Π_S, which is the notation we will use to represent the projection on S. Introducing the notations $A_S := \Pi_S A$, $B_S := \Pi_S B$ and bearing in mind that $\Pi_S K = K_S$ is deduced from Hypothesis 1, we obtain for $t \geqslant 0$:

$$\begin{cases} \mathbf{s}'_\varepsilon(t) = \overline{A}\mathbf{s}_\varepsilon(t) + \overline{B}\mathbf{s}_\varepsilon(t-r) + \mathcal{V}^* A\mathbf{q}_\varepsilon(t) + \mathcal{V}^* B\mathbf{q}_\varepsilon(t-r), \\ \mathbf{q}'_\varepsilon(t) = A_S\mathcal{V}\mathbf{s}_\varepsilon(t) + B_S\mathcal{V}\mathbf{s}_\varepsilon(t-r) + A_S\mathbf{q}_\varepsilon(t) + B_S\mathbf{q}_\varepsilon(t-r) + \frac{1}{\varepsilon}K_S\mathbf{q}_\varepsilon(t), \end{cases} \tag{8}$$

together with the initial condition

$$\mathbf{s}_\varepsilon(t) = \psi(t), \quad \mathbf{q}_\varepsilon(t) = \varphi(t), \quad t \in [-r, 0], \tag{9}$$

where the initial data $\Phi \in C([-r, 0]; \mathbf{R}^N)$ is decomposed by $\Phi = \mathcal{V}\psi + \varphi$.

3.1. Expression of $\mathbf{q}_\varepsilon$ *in terms of* $\mathbf{s}_\varepsilon$

First of all we solve the second equation of system (8) which permits $\mathbf{q}_\varepsilon$ to be expressed in terms of $\mathbf{s}_\varepsilon$.

To this end, we start by considering the associated non-homogeneous problem, with $\mathbf{f} \in C^0([0, +\infty); S)$:

$$\begin{cases} \mathbf{q}'(t) = \frac{1}{\varepsilon}K_S\mathbf{q}(t) + A_S\mathbf{q}(t) + B_S\mathbf{q}(t-r) + \mathbf{f}(t), & t \geqslant 0, \\ \mathbf{q}(t) = \varphi(t), & t \in [-r, 0] \end{cases}$$

whose solution can be expressed through a *variation of constants formula*, for $t \geqslant 0$:

$$\mathbf{q}(t) = \mathcal{W}_\varepsilon(t)\varphi(0) + \int_{-r}^{0} \mathcal{W}_\varepsilon(t-r-\sigma)B_S\varphi(\sigma)\,d\sigma + \int_0^t \mathcal{W}_\varepsilon(t-\sigma)\mathbf{f}(\sigma)\,d\sigma, \tag{10}$$

where $\mathcal{W}_\varepsilon$ is the so-called *fundamental solution*. It is known from the general theory (see [10]) that $\mathcal{W}_\varepsilon : [-r, +\infty) \to \mathcal{L}(S)$ is the unique solution to the problem

$$\begin{cases} \mathcal{W}'_\varepsilon(t) = \frac{1}{\varepsilon}K_S\mathcal{W}_\varepsilon(t) + A_S\mathcal{W}_\varepsilon(t) + B_S\mathcal{W}_\varepsilon(t-r), \quad t \geqslant 0, \\ \mathcal{W}_\varepsilon(t) = 0, \quad t \in [-r, 0); \quad \mathcal{W}_\varepsilon(0) = I_S \end{cases} \tag{11}$$

together with $\mathcal{W}_\varepsilon \in C^0([0, +\infty); \mathcal{L}(S))$.

Lemma 1. *Under Hypothesis* 1*, there exist three constants* $a > 0$*,* $k > 0$ *and* $\varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0)$*, the following estimation holds*:

$$\forall t \geqslant 0, \quad \|\mathcal{W}_\varepsilon(t)\| \leqslant e^{-at/\varepsilon} + k\varepsilon.$$

Proof. Using well-known results about perturbations of eigenvalues of linear operators (see [9,22]), and bearing in mind (3), we can assure the existence of $\varepsilon_0 > 0$ and $\alpha' > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, $\sigma(K_{S\varepsilon}) \subset \{z \in \mathbf{C}; \operatorname{Re} z < -\alpha'\}$ where $K_{S\varepsilon} := K_S + \varepsilon A_S$, hence an equivalent matrix norm can be chosen for which there exists $a > 0$ such that for each $\varepsilon > 0$: $\forall t \geqslant 0$, $\|e^{(t/\varepsilon)K_S}\| \leqslant e^{-at/\varepsilon}$ (see [25,26]).

With the aim of obtaining an estimation to the fundamental solution $\mathcal{W}_\varepsilon(t)$, we will solve Eq. (11) step-by-step.

For $t \in [0, r]$:

$$\mathcal{W}_\varepsilon'(t) = \frac{1}{\varepsilon} K_{S\varepsilon} \mathcal{W}_\varepsilon(t), \qquad \mathcal{W}_\varepsilon(0) = I_S$$

and hence

$$\mathcal{W}_\varepsilon(t) = e^{(t/\varepsilon)K_{S\varepsilon}} \quad \Rightarrow \quad \|\mathcal{W}_\varepsilon(t)\| \leqslant e^{-at/\varepsilon}.$$

If $t \in [r, 2r]$, (11) becomes the Cauchy problem:

$$\mathcal{W}_\varepsilon'(t) = \frac{1}{\varepsilon} K_{S\varepsilon} \mathcal{W}_\varepsilon(t) + B_S e^{((t-r)/\varepsilon)K_{S\varepsilon}}, \qquad \mathcal{W}_\varepsilon(r) = e^{(r/\varepsilon)K_{S\varepsilon}}$$

whose solution is

$$\mathcal{W}_\varepsilon(t) = e^{(t/\varepsilon)K_{S\varepsilon}} + \int_r^t e^{((t-\sigma)/\varepsilon)K_{S\varepsilon}} B_S e^{((\sigma-r)/\varepsilon)K_{S\varepsilon}}\, d\sigma$$

so that

$$\|\mathcal{W}_\varepsilon(t)\| \leqslant e^{-at/\varepsilon} + \|B_S\| e^{-a(t-r)/\varepsilon}(t-r).$$

Proceeding this way it is easy to obtain $\forall t \in [kr, (k+1)r]$:

$$\|\mathcal{W}_\varepsilon(t)\| \leqslant e^{-at/\varepsilon} + \sum_{j=1}^{k} \|B_S\|^j \frac{(t-jr)^j}{j!} e^{-a(t-jr)/\varepsilon} \leqslant e^{-at/\varepsilon} + \sum_{j=1}^{k} \frac{\varepsilon^j \|B_S\|^j}{a^j},$$

where we have used the estimation

$$\frac{(t-jr)^j}{j!} = \frac{\varepsilon^j}{a^j} \frac{[(a/\varepsilon)(t-jr)]^j}{j!} \leqslant \frac{\varepsilon^j}{a^j} e^{a(t-jr)/\varepsilon}.$$

Choosing $\varepsilon_0 \in (0, a/(2\|B_S\|))$, it holds that $\forall \varepsilon \in (0, \varepsilon_0)$ and $\forall t \geqslant 0$:

$$\|\mathcal{W}_\varepsilon(t)\| \leqslant e^{-at/\varepsilon} + k\varepsilon; \quad k := \frac{2\|B_S\|}{a}.$$

The lemma is proved. □

The variation of constants formula (10) applied to the second equation of system (8) provides:

$$\mathbf{q}_\varepsilon(t) = \mathcal{W}_\varepsilon(t)\varphi(0) + \int_{-r}^{0} \mathcal{W}_\varepsilon(t-r-\sigma)B_S\varphi(\sigma)\,d\sigma + \int_{0}^{t} \mathcal{W}_\varepsilon(t-\sigma)B_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma-r)\,d\sigma + \int_{0}^{t} \mathcal{W}_\varepsilon(t-\sigma)A_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma)\,d\sigma. \tag{12}$$

3.2. *Formulation of a fixed point problem for* $\mathbf{s}_\varepsilon$

Before substituting (12) into the first equation of (8), we make certain transformations in (12) that will help us to simplify calculations. Let us notice, firstly, that for $t \geqslant 0$ we can write:

$$\int_{0}^{t} \mathbf{s}_\varepsilon(\sigma-r)\,d\sigma = \int_{-r}^{0} K(t,\sigma)\psi(\sigma)\,d\sigma + \int_{0}^{t} G(t,\sigma)\mathbf{s}_\varepsilon(\sigma)\,d\sigma,$$
$$\int_{0}^{t} \mathbf{q}_\varepsilon(\sigma-r)\,d\sigma = \int_{-r}^{0} K(t,\sigma)\varphi(\sigma)\,d\sigma + \int_{0}^{t} G(t,\sigma)\mathbf{q}_\varepsilon(\sigma)\,d\sigma,$$

where we have introduced the notations

$$K(t,\sigma) := H(r-t)H(t-r-\sigma) + H(t-r),$$
$$G(t,\sigma) := H(t-r)H(t-r-\sigma),$$

$H(\cdot)$ being the Heaviside function (i.e., $H(u) = 1$ if $u > 0$, $H(u) = 0$ if $u < 0$).

Then, the integral form of the first equation of system (12) can be written, for $t \geqslant 0$ as

$$\mathbf{s}_\varepsilon(t) = \psi(0) + \mathcal{V}^* B \int_{-r}^{0} K(t,\sigma)\Phi(\sigma)\,d\sigma + \int_{0}^{t} \overline{L}(t,\sigma)\mathbf{s}_\varepsilon(\sigma)\,d\sigma + \mathcal{V}^* \int_{0}^{t} L(t,\sigma)\mathbf{q}_\varepsilon(\sigma)\,d\sigma \tag{13}$$

with

$$\overline{L}(t,\sigma) := \overline{A} + \overline{B}G(t,\sigma), \qquad L(t,\sigma) := A + BG(t,\sigma).$$

Also, we have

$$\int_{0}^{t} \mathcal{W}_\varepsilon(t-\sigma)B_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma-r)\,d\sigma$$
$$= \int_{-r}^{t-r} \mathcal{W}_\varepsilon(t-\sigma-r)B_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma)\,d\sigma$$
$$= \int_{-r}^{0} \mathcal{W}_\varepsilon(t-r-\sigma)B_S\mathcal{V}\psi(\sigma)\,d\sigma + \int_{0}^{t} G(t,\sigma)\mathcal{W}_\varepsilon(t-r-\sigma)B_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma)\,d\sigma,$$

which transforms (12) in

$$\mathbf{q}_\varepsilon(t) = \mathcal{W}_\varepsilon(t)\varphi(0) + \int_{-r}^{0} \mathcal{W}_\varepsilon(t-r-\sigma)B_S\Phi(\sigma)\,d\sigma + \int_{0}^{t} R_\varepsilon(t,\sigma)\mathbf{s}_\varepsilon(\sigma)\,d\sigma \tag{14}$$

where

$$R_\varepsilon(t,\sigma) := \big[\mathcal{W}_\varepsilon(t-\sigma)A_S + G(t,\sigma)\mathcal{W}_\varepsilon(t-r-\sigma)B_S\big]\mathcal{V}.$$

Introducing (14) into (13), we obtain the following equation for $\mathbf{s}_\varepsilon(t)$, $t \geqslant 0$:

$$\begin{aligned}\mathbf{s}_\varepsilon(t) = {} & \psi(0) + \mathcal{V}^* B \int_{-r}^{0} K(t,\sigma)\Phi(\sigma)\,d\sigma + \int_{0}^{t} \overline{L}(t,\sigma)\mathbf{s}_\varepsilon(\sigma)\,d\sigma \\ & + \mathcal{V}^* \int_{0}^{t} L(t,\sigma)\mathcal{W}_\varepsilon(\sigma)\varphi(0)\,d\sigma \\ & + \mathcal{V}^* \int_{0}^{t} L(t,w)\left(\int_{-r}^{0} \mathcal{W}_\varepsilon(w-r-\sigma)B_S\Phi(\sigma)\,d\sigma\right)dw \\ & + \mathcal{V}^* \int_{0}^{t} L(t,w)\left(\int_{0}^{w} R_\varepsilon(w,\sigma)\mathbf{s}_\varepsilon(\sigma)\,d\sigma\right)dw.\end{aligned} \tag{15}$$

This equation will be formulated below as a fixed point problem for certain operators on a Banach space.

First of all, we consider for each $\gamma > 0$, the set

$$E_\gamma := \Big\{\mathbf{f} \in C\big([0,+\infty)\ \mathbf{R}^q\big);\ \|\mathbf{f}\|_\gamma := \sup_{t\geqslant 0} e^{-\gamma t}\big\|\mathbf{f}(t)\big\| < +\infty\Big\}$$

which is a Banach space with respect to the norm $\|\cdot\|_\gamma$.

We introduce the following operators:

Operator $\mathcal{H}_0$. This is the operator (independent of ε), $\mathcal{H}_0 : E_\gamma \to E_\gamma$ defined by

$$\mathcal{H}_0(\mathbf{f})(t) := \int_{0}^{t} \overline{L}(t,\sigma)\mathbf{f}(\sigma)\,d\sigma.$$

Lemma 2. *For each $\gamma > 0$, $\mathcal{H}_0$ is a bounded linear operator in E_γ. Moreover, there exists $\gamma_0 > 0$ such that $\forall \gamma \geqslant \gamma_0$, $\mathcal{H}_0$ is a strict contraction in E_γ.*

Proof. Let $\mathbf{f} \in E_\gamma$. Then

$$\big\|\mathcal{H}_0(\mathbf{f})\big\|_\gamma \leqslant \|\mathbf{f}\|_\gamma \sup_{t\geqslant 0} e^{-\gamma t} \int_{0}^{t} \big[\|\overline{A}\| + \|\overline{B}\|\big|G(t,\sigma)\big|\big]e^{\gamma\sigma}\,d\sigma \leqslant \frac{\|\overline{A}\| + \|\overline{B}\|}{\gamma}\|\mathbf{f}\|_\gamma.$$

This estimation shows that $\mathcal{H}_0$ is a bounded linear operator in E_γ for each $\gamma > 0$. Choosing γ_0 such that $\gamma_0 > \|\overline{A}\| + \|\overline{B}\|$, it is clear that $\forall\gamma \geqslant \gamma_0$, $\mathcal{H}_0$ is a strict contraction in E_γ. The lemma is proved. □

Operator $\mathcal{A}_\varepsilon$. This is the operator $\mathcal{A}_\varepsilon : E_\gamma \to E_\gamma$ defined by

$$\mathcal{A}_\varepsilon(\mathbf{f})(t) := \mathcal{V}^* \int_0^t L(t,w)\left(\int_0^w R_\varepsilon(w,\sigma)\mathbf{f}(\sigma)\,d\sigma\right)dw.$$

Lemma 3. *For each $\gamma > 0$ and $0 < \varepsilon < \varepsilon_0$, $\mathcal{A}_\varepsilon$ is a bounded linear operator in E_γ and there exists a constant $C_1(\gamma) > 0$ such that*

$$\|\mathcal{A}_\varepsilon\| \leqslant \varepsilon C_1(\gamma).$$

Proof. Let $\mathbf{f} \in E_\gamma$. Since

$$\begin{aligned}
&\big\|L(t,w)\big\| \leqslant \|A\| + \|B\|\big|G(t,w)\big| \leqslant \|A\| + \|B\|,\\
&\big\|R_\varepsilon(w,\sigma)\big\| \leqslant \|A_S\|\big\|\mathcal{W}_\varepsilon(w-\sigma)\big\| + \|B_S\|\big\|\mathcal{W}_\varepsilon(w-r-\sigma)\big\|,
\end{aligned}$$

we have, bearing in mind Lemma 1,

$$\begin{aligned}
\big\|\mathcal{A}_\varepsilon(\mathbf{f})\big\|_\gamma &\leqslant \|\mathcal{V}^*\|\big(\|A\| + \|B\|\big)\|\mathbf{f}\|_\gamma\Bigg[\|A_S\|\sup_{t\geqslant 0} e^{-\gamma t}\int_0^t\left(\int_0^w e^{\gamma\sigma}\big\|\mathcal{W}_\varepsilon(w-\sigma)\big\|\,d\sigma\right)dw\\
&\qquad + \|B_S\|\sup_{t\geqslant 0} e^{-\gamma t}\int_0^t\left(\int_0^w e^{\gamma\sigma}\big\|\mathcal{W}_\varepsilon(w-r-\sigma)\big\|\,d\sigma\right)dw\Bigg]\\
&\leqslant \varepsilon C_1(\gamma)\|\mathbf{f}\|_\gamma
\end{aligned}$$

where

$$C_1(\gamma) := \|\mathcal{V}^*\|\big(\|A\| + \|B\|\big)\big(\|A_S\| + \|B_S\|e^{-\gamma r}\big)\frac{1}{\gamma}\left(\frac{k}{\gamma} + \frac{1}{a}\right).$$

The lemma is proved. □

Operator $\mathcal{B}_\varepsilon$. This is the operator $\mathcal{B}_\varepsilon : C([-r,0];\mathbf{R}^N) \to E_\gamma$ defined by

$$\begin{aligned}
\mathcal{B}_\varepsilon(\Phi)(t) := \mathcal{V}^*\int_0^t L(t,\sigma)\mathcal{W}_\varepsilon(\sigma)\varphi(0)\,d\sigma\\
+ \int_0^t L(t,w)\left(\int_{-r}^0 \mathcal{W}_\varepsilon(w-r-\sigma)B_S\Phi(\sigma)\,d\sigma\right)dw.
\end{aligned}$$

Operator $\mathcal{F}_0$. This is the operator $\mathcal{F}_0 : C([-r,0];\mathbf{R}^N) \to E_\gamma$ defined by

$$\mathcal{F}_0(\Phi)(t) := \psi(0) + \mathcal{V}^* B \int_{-r}^{0} K(t,\sigma)\Phi(\sigma)\,d\sigma.$$

Bearing in mind Lemma 1, straightforward calculations similar to the ones before provide estimations for these operators, which are established without proof in the following lemma:

Lemma 4. *For each* $\gamma > 0$, $0 < \varepsilon < \varepsilon_0$, $\mathcal{B}_\varepsilon$, $\mathcal{F}_0$ *are bounded linear operators and there exist two constants* $C_i(\gamma)$, $i = 2, 3$, *such that*

$$\|\mathcal{B}_\varepsilon\| \leqslant C_2(\gamma)\varepsilon, \qquad \|\mathcal{F}_0\| \leqslant C_3(\gamma).$$

Using the previous operators, Eq. (15) is written as

$$(Id - \mathcal{H}_0 - \mathcal{A}_\varepsilon)(\mathbf{s}_\varepsilon) = \mathcal{F}_0(\Phi) + \mathcal{B}_\varepsilon(\Phi). \tag{16}$$

3.3. Convergence result

Lemma 3 provides, for $\varepsilon > 0$ small enough:

$$\|\mathcal{A}_\varepsilon\| \leqslant \varepsilon C_1(\gamma) < H_0^{-1} \quad \text{with } H_0 := \big\|(\mathrm{Id} - \mathcal{H}_0)^{-1}\big\|$$

which allows us to assure the existence of the operator $(\mathrm{Id} - \mathcal{H}_0 - \mathcal{A}_\varepsilon)^{-1}$, which can be written as

$$(\mathrm{Id} - \mathcal{H}_0 - \mathcal{A}_\varepsilon)^{-1} = (\mathrm{Id} - \mathcal{H}_0)^{-1} + \Gamma_\varepsilon,$$
$$\Gamma_\varepsilon := \left(\sum_{j=1}^{\infty} \big[(\mathrm{Id} - \mathcal{H}_0)^{-1}\mathcal{A}_\varepsilon\big]^j\right)(\mathrm{Id} - \mathcal{H}_0)^{-1}.$$

Since for $0 < \varepsilon < 1/(2H_0C_1(\gamma))$ we have

$$\|\Gamma_\varepsilon\| \leqslant H_0 \sum_{j=1}^{\infty} H_0^j \|\mathcal{A}_\varepsilon\|^j \leqslant H_0 \frac{H_0\|\mathcal{A}_\varepsilon\|}{1 - H_0\|\mathcal{A}_\varepsilon\|} \leqslant 2H_0^2\|\mathcal{A}_\varepsilon\|,$$

the following estimation holds:

$$\|\Gamma_\varepsilon\| \leqslant \varepsilon C_5(\gamma), \quad C_5(\gamma) := 2H_0^2 C_1(\gamma).$$

Therefore, Eq. (16) can be solved, giving

$$\mathbf{s}_\varepsilon = (\mathrm{Id} - \mathcal{H}_0)^{-1}\big[\mathcal{F}_0(\Phi)\big] + (\mathrm{Id} - \mathcal{H}_0)^{-1}\big[\mathcal{B}_\varepsilon(\Phi)\big] + \Gamma_\varepsilon\big(\mathcal{F}_0(\Phi) + \mathcal{B}_\varepsilon(\Phi)\big).$$

Let us define

$$\mathbf{s}_0 := (\mathrm{Id} - \mathcal{H}_0)^{-1}\big(\mathcal{F}_0(\Phi)\big)$$

which yields the following estimation:

$$\|\mathbf{s}_\varepsilon - \mathbf{s}_0\|_\gamma \leqslant \varepsilon M^*(\gamma)\|\Phi\|_C \tag{17}$$

for some constant $M^*(\gamma) > 0$, whose exact value is not relevant.

Furthermore, $\mathbf{s}_0$ satisfies $(\mathrm{Id} - \mathcal{H}_0)(\mathbf{s}_0) = \mathcal{F}_0(\Phi)$ or, in integral form for $t \geqslant 0$:

$$\mathbf{s}_0(t) = \psi(0) + \overline{A}\int_0^t \mathbf{s}_0(\sigma)\,d\sigma + \overline{B}\int_0^t G(t,\sigma)\mathbf{s}_0(\sigma)\,d\sigma + \mathcal{V}^* B\int_{-r}^0 K(t,\sigma)\Phi(\sigma)\,d\sigma.$$

Then, we can write:

(i) for $t \in [0, r]$:

$$\mathbf{s}_0(t) = \Psi(0) + \overline{A}\int_0^t \mathbf{s}_0(\sigma)\,d\sigma + \mathcal{V}^* B\int_0^t \Phi(\sigma - r)\,d\sigma;$$

(ii) for $t \geqslant 0$:

$$\mathbf{s}_0(t) = \mathbf{s}_0(r) + \overline{A}\int_r^t \mathbf{s}_0(\sigma)\,d\sigma + \overline{B}\int_r^t \mathbf{s}_0(\sigma - r)\,d\sigma,$$

which means that $\mathbf{s}_0$ satisfies the aggregated problem (5)–(6).

We are now able to prove one of the two main results of this paper, which is established in the following theorem.

Theorem 1. *Under Hypothesis* 1*, for each initial data* $\Phi \in C([-r, 0]; \mathbf{R}^N)$*,* $\Phi = \mathcal{V}\psi + \varphi$*, the corresponding solution* X_ε *to system* (1) *can be written as*

$$\forall t \geqslant r, \quad X_\varepsilon(t) = \mathcal{V}\mathbf{s}_0(t) + \mathbf{r}_\varepsilon(t), \tag{18}$$

where $\mathbf{s}_0$ *is the solution to the aggregated system* (5) *for* $t \geqslant r$*, with the initial data defined by*

$$\forall t \in [0, r], \quad \mathbf{s}_0(t) = e^{t\overline{A}}\left[\mathcal{V}^*\Phi(0) + \int_0^t e^{-\sigma\overline{A}}\mathcal{V}^* B\Phi(\sigma - r)\,d\sigma\right].$$

Moreover, there exist three constants $C > 0$*,* $C^* > 0$*,* $\gamma > 0$*, such that*

$$\forall t \geqslant r, \quad \left\|\mathbf{r}_\varepsilon(t)\right\| \leqslant \varepsilon\left(C + C^* e^{\gamma t}\right)\|\Phi\|_C. \tag{19}$$

Therefore, for each $T > r$*,* $\lim_{\varepsilon \to 0_+} X_\varepsilon = \mathcal{V}\mathbf{s}_0$ *uniformly in the interval* $[r, T]$*.*

Proof. The solution to system (1) can be written as

$$X_\varepsilon(t) = \mathcal{V}\mathbf{s}_\varepsilon(t) + \mathbf{q}_\varepsilon(t) = \mathcal{V}\mathbf{s}_0(t) + \mathcal{V}\left[\mathbf{s}_\varepsilon(t) - \mathbf{s}_0(t)\right] + \mathbf{q}_\varepsilon(t)$$

together with

$$\mathbf{q}_\varepsilon(t) = \mathcal{W}_\varepsilon(t)\varphi(0) + \int_0^t \mathcal{W}_\varepsilon(t-\sigma)A_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma)\,d\sigma + \int_0^t \mathcal{W}_\varepsilon(t-\sigma)B_S\mathcal{V}\mathbf{s}_\varepsilon(\sigma - r)\,d\sigma$$
$$+ \int_{-r}^0 \mathcal{W}_\varepsilon(t - r - \sigma)B_S\varphi(\sigma)\,d\sigma.$$

Let us define

$$\mathbf{r}_\varepsilon(t) := \mathcal{V}\big[\mathbf{s}_\varepsilon(t) - \mathrm{s}_0(t)\big] + \mathbf{q}_\varepsilon(t).$$

The proof of the theorem reduces to estimate $\mathbf{r}_\varepsilon(t)$ for $t \geqslant r$. To this end, in what follows, we denote by k_i, $i = 1, 2, \dots$, the different constants which appear in the calculations and whose specific values are not relevant.

(a) From (17), we have

$$\big\|\mathcal{V}\big(\mathbf{s}_\varepsilon(t) - \mathbf{s}_0(t)\big)\big\| \leqslant k_1 e^{\gamma t}\|\mathbf{s}_\varepsilon - \mathbf{s}_0\|_\gamma \leqslant \varepsilon k_2 e^{\gamma t}\|\Phi\|_C.$$

(b) Bearing in mind Lemma 1, for $\varepsilon > 0$ small enough and $t \geqslant r$, straightforward calculations lead to

$$\big\|\mathbf{q}_\varepsilon(t)\big\| \leqslant \varepsilon\|\Phi\|_C\big(k_3 + k_4 e^{\gamma t}\big).$$

Estimation (19) is now obtained directly from (a) and (b). This finishes the proof of the theorem. □

4. Aggregation of variables in linear functional delayed differential equations

Keeping the notations of Section 2, let us generalize the above results to the following *perturbed system* of linear delayed differential equations, depending on a small parameter $\varepsilon > 0$:

$$\begin{cases} X'(t) = L(X_t) + (1/\varepsilon)KX(t), & t > 0, \\ X_0 = \Phi \in C([-r, 0]; \mathbf{R}^N) \end{cases} \tag{20}$$

where $L : C([-r, 0]; \mathbf{R}^N) \to \mathbf{R}^N$ is a bounded linear operator and X_t $(t \geqslant 0)$, is the *section of function* X *at time* t, namely, $X_t(\theta) := X(t + \theta)$, $\theta \in [-r, 0]$.

We will formulate system (20) in the framework of semigroup theory applied to delayed differential equations, which can be seen in [18] and [19].

Defining the following bounded linear operator

$$\mathcal{L}_\varepsilon : C\big([-r, 0]; \mathbf{R}^N\big) \to \mathbf{R}^N, \qquad \mathcal{L}_\varepsilon(\Phi) := L(\Phi) + (1/\varepsilon)K\Phi(0)$$

system (20) reads as

$$X'(t) = \mathcal{L}_\varepsilon(X_t), \quad t > 0, \qquad X_0 = \Phi.$$

This is a well-known linear delayed differential equation, whose solutions define a C_0-semigroup $\{T_\varepsilon(t)\}_{t \geqslant 0}$ on $C([-r, 0]; \mathbf{R}^N)$ in such a way that for each $\Phi \in C([-r, 0]; \mathbf{R}^N)$, $(X_\varepsilon)_t := T_\varepsilon(t)\Phi$ is the unique solution to the Cauchy problem (20) corresponding to the initial value $(X_\varepsilon)_0 = \Phi$.

4.1. The aggregated model

Assuming Hypothesis 1, the *aggregated variables*: $\mathbf{s}(t) := \mathcal{V}^* X(t)$ satisfy the system

$$\mathbf{s}'(t) = \mathcal{V}^* L(\mathcal{V}\mathbf{s}_t) + \mathcal{V}^* L\big((X_S)_t\big).$$

We propose as *aggregated model* the following approximated system with the aggregated variables as the unique state variables

$$\mathbf{s}'(t) = \overline{L}(\mathbf{s}_t), \quad t \geqslant r, \tag{21}$$

where $\overline{L}$ is the linear bounded operator defined by

$$\overline{L}: C([-r,0];\mathbf{R}^q) \to \mathbf{R}^q, \qquad \overline{L}(\psi) := \mathcal{V}^* L(\mathcal{V}\psi).$$

This system is a delayed linear differential system of equations to which the general semigroup theory of [18,19] can also be applied.

As in Section 2, the initial data in $[0,r]$ should be constructed, but in this abstract setting it presents higher mathematical difficulties. In particular, we should use the Riesz representation theorem of bounded linear operators on $C([-r,0];\mathbf{R}^N)$ (see [8]). Operator L can be written as a Riemann–Stieltjes integral:

$$\forall \Phi \in C([-r,0];\mathbf{R}^N), \qquad L(\Phi) = \int_{[-r,0]} [d\eta(\theta)]\Phi(\theta) \tag{22}$$

where $\eta(\theta)$ is a bounded variation $N \times N$ matrix-valued function. The well-known properties of Riemann–Stieltjes integrals (see [1]) allow us to assure the existence of the following integral, for each $t \in (0,r)$:

$$I(t,\Phi) := \int_{[-r,-t]} [d\eta(\theta)]\Phi(t+\theta).$$

In what follows we will show that the contribution of sections of the initial data $\Phi = \mathcal{V}\psi + \varphi$ to the aggregated model in $[0,r]$ is given by $I(t,\varphi)$.

4.2. Comparison between the solutions to systems (20) and (21)

As in Section 3, we decompose the solution to (20) according to the direct sum decomposition (2). Projecting the equation onto the subspaces $\ker K$ and S and introducing the notation $L_S := \Pi_S L$, we obtain for $t \geqslant 0$:

$$\begin{cases} \mathbf{s}'_\varepsilon(t) = \overline{L}((\mathbf{s}_\varepsilon)_t) + \mathcal{V}^* L((\mathbf{q}_\varepsilon)_t), \\ \mathbf{q}'_\varepsilon(t) = L_S(\mathcal{V}(\mathbf{s}_\varepsilon)_t) + L_S((\mathbf{q}_\varepsilon)_t) + \dfrac{1}{\varepsilon} K_S \mathbf{q}_\varepsilon(t) \end{cases} \tag{23}$$

together with the initial condition

$$(\mathbf{s}_\varepsilon)_0 = \psi \in C([-r,0];\mathbf{R}^q), \qquad (\mathbf{q}_\varepsilon)_0 = \varphi \in C([-r,0];S)$$

where the initial data $\Phi \in C([-r,0];\mathbf{R}^N)$ has been decomposed by $\Phi = \mathcal{V}\psi + \varphi$.

To express $\mathbf{q}_\varepsilon$ in terms of $\mathbf{s}_\varepsilon$, we start by considering the associated homogeneous problem:

$$\mathbf{q}'(t) = L_S(\mathbf{q}_t) + \frac{1}{\varepsilon} K_S \mathbf{q}(t). \tag{24}$$

Introducing the following linear bounded operator

$$\mathcal{L}_S^\varepsilon : C([-r,0];S) \to S, \qquad \mathcal{L}_S^\varepsilon(\varphi) := L_S(\varphi) + \frac{1}{\varepsilon} K_S \varphi(0),$$

Eq. (24) can be written as a delayed linear differential equation

$$\mathbf{q}'(t) = \mathcal{L}_S^\varepsilon(\mathbf{q}_t), \quad t > 0, \tag{25}$$

whose solutions define a C_0-semigroup $\{T_S^\varepsilon(t)\}_{t \geqslant 0}$ on $C([-r,0];S)$ so that

$$\mathbf{q}_t^* = T_S^\varepsilon(t)\varphi, \quad t > 0, \tag{26}$$

is the solution to (25) corresponding to the initial value $\mathbf{q}_0^* = \varphi$.

It is also known from the general theory [2] that a *fundamental solution* $\mathcal{U}_S^\varepsilon \in C(\mathbf{R}_+; \mathcal{L}(S))$ can be associated to Eq. (25), so that a *variation of constants formula* can be written for the solutions to the non-homogeneous problem. Such formula applied to our case gives

$$(\mathbf{q}_\varepsilon)_t = T_S^\varepsilon(t)\varphi + \int_0^t (\mathcal{U}_S^\varepsilon)_{t-\sigma} \otimes L_S(\mathcal{V}(\mathbf{s}_\varepsilon)_\sigma)\, d\sigma, \quad t \geqslant 0, \tag{27}$$

where $\forall \mathbf{b} \in S$, $\forall \theta \in [-r, 0]$, $\forall t \geqslant 0$, $((\mathcal{U}_S^\varepsilon)_t \otimes \mathbf{b})(\theta) := (\mathcal{U}_S^\varepsilon)_t(\theta)(\mathbf{b})$.

In order to simplify the calculations, we introduce the following notations. For each $\mathbf{f} \in C([-r, +\infty); \mathbf{R}^q)$ we define

$$\mathcal{G}_\varepsilon(\mathbf{f}) : \mathbf{R}_+ \to C([-r, 0]; S), \qquad \mathcal{G}_\varepsilon(\mathbf{f})(t) := \int_0^t (\mathcal{U}_S^\varepsilon)_{t-\sigma} \otimes L_S(\mathcal{V}\mathbf{f}_\sigma)\, d\sigma.$$

With the help of this notation, (27) can be written like this

$$(\mathbf{q}_\varepsilon)_t = T_S^\varepsilon(t)\varphi + \mathcal{G}_\varepsilon(\mathbf{s}_\varepsilon)(t), \quad t \geqslant 0.$$

This expression, substituted into the first equation of (23), provides the following equation for $\mathbf{s}_\varepsilon$, $t \geqslant 0$:

$$\mathbf{s}_\varepsilon(t) = \psi(0) + \int_0^t \overline{L}((\mathbf{s}_\varepsilon)_\sigma)\, d\sigma + \mathcal{V}^* \int_0^t L(T_S^\varepsilon(\sigma)\varphi)\, d\sigma + \mathcal{V}^* \int_0^t L(\mathcal{G}_\varepsilon(\mathbf{s}_\varepsilon)(\sigma))\, d\sigma. \tag{28}$$

For each initial condition $\psi \in C([-r, 0]; \mathbf{R}^q)$, we define the function $\tilde{\psi} : [-r, +\infty) \to \mathbf{R}^q$ by

$$\tilde{\psi}(t) := \begin{cases} \psi(t), & t \in [-r, 0], \\ \psi(0), & t \geqslant 0, \end{cases}$$

and we make in Eq. (28) the change of unknown function defined by

$$\mathbf{s}_\varepsilon(t) := \mathbf{y}_\varepsilon(t) + \tilde{\psi}(t), \quad t \in [-r, +\infty).$$

We obtain, for $t \geqslant 0$:

$$\mathbf{y}_\varepsilon(t) = \int_0^t \overline{L}((\mathbf{y}_\varepsilon)_\sigma)\, d\sigma + \mathcal{V}^* \int_0^t L(\mathcal{G}_\varepsilon(\mathbf{y}_\varepsilon)(\sigma))\, d\sigma + \int_0^t \overline{L}(\tilde{\psi}_\sigma)\, d\sigma + \mathcal{V}^* \int_0^t L(T_S^\varepsilon(\sigma)\varphi)\, d\sigma + \mathcal{V}^* \int_0^t L(\mathcal{G}_\varepsilon(\tilde{\psi})(\sigma))\, d\sigma \tag{29}$$

together with $\mathbf{y}_\varepsilon(t) = \mathbf{0}$, $t \in [-r, 0]$.

In what follows we will formulate this equation as an abstract fixed problem for some operators in a Banach space.

For each $\gamma > 0$, let E_γ be the set

$$E_\gamma := \left\{ \mathbf{f} \in C([-r, +\infty); \mathbf{R}^q);\ \begin{array}{l} \mathbf{f}(\theta) = 0,\ \theta \in [-r, 0], \\ \|\mathbf{f}\|_\gamma := \sup_{t \geqslant 0} e^{-\gamma t} \|\mathbf{f}(t)\| < +\infty \end{array} \right\}$$

which is a Banach space endowed with the norm $\|\cdot\|_\gamma$.

Let us define the following operators:

Operator $\mathcal{H}_0$. This is the operator $\mathcal{H}_0 : E_\gamma \to E_\gamma$ defined by

$$\forall \mathbf{f} \in E_\gamma, \quad \mathcal{H}_0(\mathbf{f})(t) := \begin{cases} \int_0^t \overline{L}(\mathbf{f}_\sigma)\, d\sigma, & t \geqslant 0, \\ \mathbf{0}, & t \in [-r, 0]. \end{cases}$$

Straightforward calculations show that $\|\mathcal{H}_0\| \leqslant \|\overline{L}\|/\gamma$ and hence Lemma 2 remains valid.

Operator $\mathcal{A}_\varepsilon$. This is the operator $\mathcal{A}_\varepsilon : E_\gamma \to E_\gamma$ defined by

$$\forall \mathbf{f} \in E_\gamma, \quad \mathcal{A}_\varepsilon(\mathbf{f})(t) := \begin{cases} \mathcal{V}^* \int_0^t L(\mathcal{G}_\varepsilon(\mathbf{f})(\sigma)\, d\sigma, & t \geqslant 0, \\ \mathbf{0}, & t \in [-r, 0]. \end{cases}$$

Hypothesis 2. *There exist two constants $a > 0$, $k > 0$ for which the solution* (26) *to the homogeneous problem* (25) *satisfies the estimation*

$$\left\| \mathbf{q}_\varepsilon^*(t) \right\| \leqslant \left(k\varepsilon + e^{-a/\varepsilon t} \right) \left\| \varphi(0) \right\|, \quad t > 0.$$

As a consequence, we have

$$\left\| T_S^\varepsilon(t) \right\| \leqslant k\varepsilon + e^{(-a/\varepsilon)(t-r)}, \quad t \geqslant r.$$

Under Hypothesis 2, straightforward calculations show that Lemma 3 remains valid.

The main difficulty for the analysis of this abstract formulation resides in the behaviour of the term $\mathcal{V}^* \int_0^t L(T_S^\varepsilon(\sigma)\varphi)\, d\sigma$ when $\varepsilon \to 0$. Using for L the representation (22), we can write

$$\begin{aligned} L\left(T_S^\varepsilon(\sigma)\varphi\right) &= \int_{[-r,-\sigma]} \left[d\eta(\theta)\right]\varphi(\sigma+\theta) + \int_{[-\sigma,0]} \left[d\eta(\theta)\right]\mathbf{q}_\varepsilon^*(\sigma+\theta) \\ &:= I(\sigma,\varphi) + J_\varepsilon(\sigma,\varphi). \end{aligned}$$

Hypothesis 3. *There exists a constant $\tilde{M} > 0$ such that*

$$\int_0^r \left\| J_\varepsilon(\sigma,\varphi) \right\| d\sigma \leqslant \varepsilon \tilde{M} \|\varphi\|_C.$$

Remark. Let us notice that the operators

$$\begin{aligned} &L(\varphi) := A\varphi(0) + B\varphi(-r), \\ &L(\varphi) := \int_{[-r,0]} M(\theta)\varphi(\theta)\, d\theta, \quad M \in L^\infty\left([-r,0]; \mathcal{L}\left(\mathbf{R}^N\right)\right) \end{aligned}$$

satisfy Hypothesis 3.

Operators $\mathcal{F}_0$, $\mathcal{B}_\varepsilon$. These are the operators $\mathcal{F}_0, \mathcal{B}_\varepsilon : C([-r,0];S) \to E_\gamma$ defined for $t \geqslant 0$ by

$$\mathcal{F}_0(\varphi)(t) := \mathcal{V}^* \int_0^{\min(t,r)} I(\sigma,\varphi)\,d\sigma,$$

$$\mathcal{B}_\varepsilon(\varphi)(t) := \mathcal{V}^* \left[\int_0^{\min(t,r)} J_\varepsilon(\sigma,\varphi)\,d\sigma + H(t-r) \int_r^t L\big(T_S^\varepsilon(\sigma)\varphi\big)\,d\sigma \right]$$

together with $\mathcal{F}_0(\varphi)(t) = \mathcal{B}_\varepsilon(\varphi)(t) = 0$, $t \in [-r,0]$.

Straightforward calculations lead to

Lemma 5. *Under Hypotheses* 2, 3, *for each $\gamma > 0$, $0 < \varepsilon < \varepsilon_0$, $\mathcal{F}_0$, $\mathcal{B}_\varepsilon$ are bounded linear operators and there exist two constants $C_i(\gamma)$, $i = 1,2$, such that*

$$\|\mathcal{F}_0\| \leqslant C_2(\gamma), \qquad \|\mathcal{B}_\varepsilon\| \leqslant \varepsilon C_3(\gamma).$$

Using the previous operators, Eq. (29) reads as

$$(\mathrm{Id} - \mathcal{H}_0 - \mathcal{A}_\varepsilon)(\mathbf{y}_\varepsilon) = (\mathcal{H}_0 + \mathcal{A}_\varepsilon)(\tilde{\psi}) + (\mathcal{F}_0 + \mathcal{B}_\varepsilon)(\varphi)$$

and then, keeping the notations of Section 3.3, calculations similar to those made in this subsection lead to

$$\mathbf{y}_\varepsilon = (\mathrm{Id} - \mathcal{H}_0)^{-1}\big[\mathcal{H}_0(\tilde{\psi}) + \mathcal{F}_0(\varphi)\big] + \Gamma_\varepsilon\big[\mathcal{H}_0(\tilde{\psi}) + \mathcal{F}_0(\varphi)\big] + (\mathrm{Id} - \mathcal{H}_0 - \mathcal{A}_\varepsilon)^{-1}\big[\mathcal{A}_\varepsilon(\tilde{\psi}) + \mathcal{B}_\varepsilon(\varphi)\big].$$

Defining

$$\mathbf{y}_0 := (\mathrm{Id} - \mathcal{H}_0)^{-1}\big[\mathcal{H}_0(\tilde{\psi}) + \mathcal{F}_0(\varphi)\big]$$

the following estimation holds:

$$\|\mathbf{y}_\varepsilon - \mathbf{y}_0\|_\gamma \leqslant \varepsilon M^*(\gamma)\|\Phi\|_C.$$

Furthermore, $\mathbf{y}_0$ satisfies

$$(\mathrm{Id} - \mathcal{H}_0)(\mathbf{y}_0) = \mathcal{H}_0(\tilde{\psi}) + \mathcal{F}_0(\varphi)$$

or, in integral form, for $t \geqslant 0$:

$$\mathbf{y}_0(t) = \int_0^t \overline{L}(\mathbf{y}_{0\sigma})\,d\sigma + \int_0^t \overline{L}(\tilde{\psi}_\sigma)\,d\sigma + \mathcal{V}^* \int_0^{\min(t,r)} I(\sigma,\varphi)\,d\sigma.$$

Then, $\mathbf{s}_0(t) := \mathbf{y}_0(t) + \tilde{\psi}(t)$ satisfies:

$$\mathbf{s}_0(t) = \psi(t), \quad t \in [-r,0], \tag{30}$$

$$\mathbf{s}_0'(t) = \overline{L}(\mathbf{s}_{0t}) + \mathcal{V}^* I(t,\varphi), \quad t \in [0,r], \tag{31}$$

$$\mathbf{s}_0'(t) = \overline{L}(\mathbf{s}_{0t}), \quad t \geqslant r,$$

or, in other words, $\mathbf{s}_0$ is the solution to the aggregated model (21) with the initial data defined by (30)–(31).

The following theorem summarizes this result of approximation, which extends Theorem 1 to the setting of linear functional differential equations. The proof is omitted, as it is a direct consequence of previous considerations.

Theorem 2. *Under Hypotheses* 1–3, *for each initial data* $\Phi \in C([-r,0];\mathbf{R}^N)$, $\Phi = \mathcal{V}\psi + \varphi$, *the corresponding solution to the system* (20) *can be written for* $t \geqslant 0$ *as*

$$X_\varepsilon(t) = \mathcal{V}\mathbf{s}_0(t) + \mathbf{r}_\varepsilon(t)$$

where $\mathbf{s}_0$ *is the solution to the aggregated system* (21) *for* $t \geqslant r$, *with the initial data defined by* (30)–(31).

Moreover, there exist three constants $C > 0$, $C^* > 0$, $\gamma > 0$ *such that*

$$\forall t \geqslant r, \quad \|\mathbf{r}_\varepsilon(t)\| \leqslant \varepsilon\big(C + C^* e^{\gamma t}\big)\|\Phi\|_C.$$

Therefore, for each $T > r$,

$$\lim_{\varepsilon \to 0_+} X_\varepsilon = \mathcal{V}\mathbf{s}_0$$

uniformly in the interval $[r, T]$.

5. Application to a structured model of population dynamics with two time scales

Let us consider a continuous-time two-stage structured model of a population living in an environment divided into two different sites [23]. Let us refer to the individuals in the two stages as juveniles and adults, so that $j_i(t)$ and $n_i(t)$ denote the juvenile and adult population respectively at site i, $i = 1,2$. Changes in the juvenile population at site i occur through birth, maturation to the adult stage and death. Therefore, in absence of migrations, the growth rate is expressed as $\beta_i n_i(t) - e^{-\mu_i^* r_i}\beta_i n_i(t - r_i) - \mu_i^* j_i(t)$ where $\beta_i, \mu_i^*, \mu_i \geqslant 0$ are the fecundities and per capita death rates of juveniles and adults respectively and $r_i > 0$ is the juvenile-stage duration in site i. Without loss of generality, we suppose $0 < r_1 < r_2$.

In a similar way, the adult population growth rate in site i must contain recruitment and mortality terms so that in absence of migrations reads $e^{-\mu_i^* r_i}\beta_i n_i(t - r_i) - \mu_i n_i(t)$.

We consider a model which includes the demographic processes described below, together with a fast migration process between sites for the adult population defined by two parameters: $m_1 > 0$ represents the migration rate from site 1 to site 2 and $m_2 > 0$ is the migration rate from site 2 to site 1.

The difference between the two time scales: slow (demography) and fast (migration) is represented by a small parameter $\varepsilon > 0$:

$$\begin{cases} j_1'(t) = \beta_1 n_1(t) - e^{-\mu_1^* r_1}\beta_1 n_1(t - r_1) - \mu_1^* j_1(t), \\ j_2'(t) = \beta_2 n_2(t) - e^{-\mu_2^* r_2}\beta_2 n_2(t - r_2) - \mu_2^* j_2(t), \\ n_1'(t) = (1/\varepsilon)[m_2 n_2(t) - m_1 n_1(t)] + e^{-\mu_1^* r_1}\beta_1 n_1(t - r_1) - \mu_1 n_1(t), \\ n_2'(t) = (1/\varepsilon)[m_1 n_1(t) - m_2 n_2(t)] + e^{-\mu_2^* r_2}\beta_2 n_2(t - r_2) - \mu_2 n_2(t). \end{cases}$$

As we notice, the last two equations of the above system are autonomous, so we can reduce the system into them:

$$\mathbf{n}'(t) = \frac{1}{\varepsilon}K\mathbf{n}(t) + A\mathbf{n}(t) + B_1\mathbf{n}(t - r_1) + B_2\mathbf{n}(t - r_2) \tag{32}$$

where

$$\mathbf{n}(t) := \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix}, \qquad K := \begin{pmatrix} -m_1 & m_2 \\ m_1 & -m_2 \end{pmatrix}, \qquad A = \begin{pmatrix} -\mu_1 & 0 \\ 0 & -\mu_2 \end{pmatrix},$$
$$B_1 := \begin{pmatrix} e^{-\mu_1^* r_1}\beta_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 0 & 0 \\ 0 & e^{-\mu_2^* r_2}\beta_2 \end{pmatrix}$$

together with an initial condition $\Phi(t) := (\Phi_1(t), \Phi_2(t))^T$, $t \in [-r_2, 0]$.

Matrix K satisfies Hypothesis 1 and in order to build the *aggregated model* of (32) we choose the right and left eigenvectors associated to eigenvalue $\lambda = 0$ of K as $\mathbf{v} := 1/(m_1 + m_2) \times (m_2, m_1)^T$, $\mathbf{v}^* := (1, 1)^T$, so that we construct an aggregated model for the total adult population:

$$n(t) := (\mathbf{v}^*)^T \mathbf{n}(t) = n_1(t) + n_2(t).$$

This model does not fit in the formulation of Section 2, due to the two different delays. Then, we apply the method explained in Section 4. We have

$$\forall \Phi \in C\big([-r_2, 0]; \mathbf{R}^2\big), \quad L(\Phi) := A\Phi(0) + B_1\Phi(-r_1) + B_2\Phi(-r_2)$$

and then, $\forall \psi \in C([-r_2, 0]; \mathbf{R})$:

$$\bar{L}(\psi) := -\mu^*\psi(0) + \nu_1^*\psi(-r_1) + \nu_2^*\psi(-r_2)$$

with

$$\mu^* := \frac{\mu_1 m_2 + \mu_2 m_1}{m_1 + m_2}, \qquad \nu_1^* := \frac{e^{-\mu_1^* r_1}\beta_1 m_2}{m_1 + m_2}, \qquad \nu_2^* := \frac{e^{-\mu_2^* r_2}\beta_2 m_1}{m_1 + m_2}.$$

The aggregated model is, for $t \geqslant r_2$:

$$n'(t) = -\mu^* n(t) + \nu_1^* n(t - r_1) + \nu_2^* n(t - r_2) \tag{33}$$

together with the initial condition defined by

$$\begin{aligned} &n(t) = \Phi_1(t) + \Phi_2(t), \quad t \in [-r_2, 0], \\ &n'(t) = -\mu^* n(t) + e^{-\mu_1^* r_1}\beta_1\Phi_1(t - r_1) + e^{-\mu_2^* r_2}\beta_2\Phi_2(t - r_2), \quad t \in [0, r_1], \\ &n'(t) = -\mu^* n(t) + \nu_1^* n(t - r_1) + e^{-\mu_2^* r_2}\beta_2\Phi_2(t - r_2), \quad t \in [r_1, r_2]. \end{aligned}$$

We have reduced the initial complete system of four equations to a single equation governing the total adult population. If the solution to this equation is given, then the juvenile population densities can be derived from it.

With the aim of applying the theoretical results obtained in Section 4, we have to prove that model (32) satisfies Hypotheses 2 and 3. In this example, $\mathbf{q}_\varepsilon^*$ is the solution to

$$\mathbf{q}'(t) = \frac{1}{\varepsilon} K_S \mathbf{q}(t) + A_S \mathbf{q}(t) + B_{1S}\mathbf{q}(t - r_1) + B_{2S}\mathbf{q}(t - r_2).$$

The estimation proposed in Hypothesis 2 can be obtained by solving step-by-step the previous equation. The calculations are similar to those done in the proof of Lemma 1, although more laborious and which are not reproduced here. From this estimation Hypothesis 3 is satisfied since

$$\int_0^{r_2} \big\| J(\sigma, \varphi) \big\| \, d\sigma \leqslant \|A\| \int_0^{r_2} \big\| \mathbf{q}_\varepsilon^*(\sigma) \big\| \, d\sigma + \int_{r_1}^{r_2} \big\| \mathbf{q}_\varepsilon^*(\sigma - r_1) \big\| \, d\sigma.$$

In conclusion, Theorem 2 assures that, for each $T > r_2$, the solution to system (32) satisfies

$$\lim_{\varepsilon\to 0_+}\begin{pmatrix} n_{1\varepsilon}(t) \\ n_{2\varepsilon}(t)\end{pmatrix} = \frac{1}{m_1+m_2}\begin{pmatrix} m_2 \\ m_1\end{pmatrix} n(t)$$

uniformly in $[r_2, T]$, $n(t)$ being the solution to the aggregated model.

An analysis of solutions to Eq. (33) can be made from a characteristic equation, which is a transcendental equation and whose study is out of the scope of this paper. Nevertheless, to present some idea about it, let us simplify our example assuming $r_1 = r_2 = r$. In this case, the associated characteristic equation is

$$\lambda = -\mu^* + \left(\nu_1^* + \nu_2^*\right)e^{-\lambda r}.$$

Introducing the notations: $z := \lambda r$, $p := -r\mu^*$, $q := r(\nu_1^* + \nu_2^*)$ the equation can be written as

$$pe^z + q - ze^z = 0 \tag{34}$$

to which the following result applies:

Theorem 3. [20] *A necessary and sufficient condition for all the roots of Eq.* (34) *to have a negative real part is that*

(i) $p < 1$ *and*
(ii) $p < -q < (\theta^2 + p^2)^{1/2}$ *where* θ *is the only root of the equation* $\theta = p\,\mathrm{tg}\,\theta$, $0 < \theta < \pi$ *for* $p \neq 0$*, or* $\theta = \pi/2$ *for* $p = 0$.

This gives the following condition for extinction of the population:

$$\left(\mu_1 - e^{-\mu_1^* r}\beta_1\right)m_2 + \left(\mu_2 - e^{-\mu_2^* r}\beta_2\right)m_1 > 0.$$

6. Conclusion

In this paper we have extended the process of reduction of a complex system with two time scales using the aggregation of variables method to the setting of linear delayed functional differential equations. A relevant conclusion is that the delay introduces some difficulties in the construction of the reduced model. The so-called quick derivation method leads to the right reduced model for $t \geqslant r$, but a careful analysis is needed for the initial data in $[0, r]$.

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