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AGGREGATION METHODS IN DISCRETE MODELS

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ABSTRACT

The aim of this work is to extend approximate aggregation methods for multi-time scale systems of ordinary differential equations to time discrete models. We give general methods in order to reduce a large scale time discrete model into an aggregated model for a few number of slow macro-variables. We study the case of linear systems. We demonstrate that the elements defining the asymptotic behaviours of the initial and aggregate models are similar to first order. We apply this method to the case of an age-structured population with sub-populations in each age classes associated to different spacial patches or different individual activities. A fast time scale is assumed for patch or activity dynamics with respect to aging and reproduction processes. Our method allows us to aggregate the system into a classical Leslie model in which the fecundity and aging parameters of the aggregated model are expressed in terms of the equilibrium proportions of individuals in the different activities or patches.

Keywords: Discrete aggregation methods, time scales, hierarchical organization, Leslie matrix.

1. Introduction

Aggregation methods allow us to simplify a system involving a large number of coupled variables. Perfect aggregation corresponds to the reduction of a large scale system of ordinary differential equations into an aggregated system involving a few number of macro-variables. Methods of perfect aggregation have been recently applied in the context of population dynamics [7]. In most cases, it is impossible to realize perfect aggregation. Indeed, in order to aggregate perfectly a differential system, it is needed that the parameters of the initial large scale system satisfy very particular conditions. As a consequence, perfect aggregation remains unsuccessful in most cases to which we would like to use it.

For this reason, approximate aggregation methods can be performed. In particular, it is possible to take advantage of the existence of different time scales in the system to realize an approximate aggregation. Perturbation methods are used and allow to aggregate the initial large scale system into an aggregated system governing slow-varying macro-variables [1].

These approximate aggregation methods have been applied in the context of population dynamics [2]. For example, one can aggregate competition or prey-predator models with many sub-populations associated to different spacial patches into a reduced system of two differential equations. This is possible when the patch dynamics is assumed to be fast with respect to the growth and the interspecific interactions.

The aim of this work is to extend approximate aggregation methods to time discrete models. In the first part, we give general results about approximate aggregation methods for discrete models in which different time scales are involved. We show that when the fast system has reached an equilibrium, the large scale system can be aggregated into a reduced system. We consider a linear model whose dynamics is governed by the strictly dominant eigenvalue and its associated eigenvector. In this contribution, we show that the dominant eigenvalues of the initial and aggregated systems are the same to the first order. It is also shown that the eigenvectors are the same to the zero order. In the second part, we give an example of an age structured population with activity dynamics. We assume a fast time scale at the individual level and a slow time scale at the population level. The aggregated system is a classical Leslie matrix model in which the fecundity and aging rates are expressed in terms of the equilibrium proportions of individuals in the different activities.

2. Variables Aggregation in a Discrete Linear Model

We suppose that we are dealing with a hierarchically organized system in the context of natural processes whose dynamics could be described in a linear and discrete way. The model consists of the next system of linear difference equations depending on the little parameter ϵ, that we call *perturbed system*

$$\begin{bmatrix} \overline{x}^1{}_{n+1}(\epsilon) \\ \vdots \\ \overline{x}^p{}_{n+1}(\epsilon) \end{bmatrix} = \begin{bmatrix} A_{11}(\epsilon) & \dots & A_{1p}(\epsilon) \\ \vdots & \ddots & \vdots \\ A_{p1}(\epsilon) & \dots & A_{pp}(\epsilon) \end{bmatrix} \begin{bmatrix} \overline{x}^1{}_n(\epsilon) \\ \vdots \\ \overline{x}^p{}_n(\epsilon) \end{bmatrix} \tag{2.1}$$

or shortly

$$X_{n+1}(\epsilon) = A(\epsilon)X_n(\epsilon)$$

where

$$\overline{x}^j{}_n = (x_n{}^{j1}, \dots, x_n{}^{jN^j})^T \ , \ j = 1, \dots, p \ , \ N = N^1 + \dots + N^p$$

stands for the number of individuals in the subgroups from 1 to N^j of the group j of the population at time n. The dynamics associated with these subgroups is being considered the fast dynamics of the system. These subgroups could represent different activities in each age class or else to different spatial patches.

Every matrix $A_{jk}(\epsilon)$ has dimensions $N^j \times N^k$, and we will suppose holomorphic dependence on ϵ.

Hypothesis (H1). The matrix $A(0)$ associated with the unperturbed system ($\epsilon = 0$) satisfies the following conditions: $A_{jk}(0) = 0$ whenever $j \neq k$, and $A_{jj}(0)$ is a regular stochastic matrix, that we will denote by P_j, $j = 1, \ldots, p$.

To describe the asymptotic properties of regular stochastic matrices (see [6]), we use the following notation:

$$P_j \overline{v}_j = \overline{v}_j; \quad {P_j}^T \overline{1}_j = \overline{1}_j; \quad \langle \overline{v}_j, \overline{1}_j \rangle = 1, \quad j = 1, \ldots, p$$

being $\overline{v}_j$ the positive eigenvector of the eigenvalue 1 of matrix P_j that verifies $\langle \overline{v}_j, \overline{1}_j \rangle = 1$, where $\overline{1}_j = (1, \ldots, 1)^T$, $(1 \times N^j)$, and $\langle *, * \rangle$ is the usual scalar product.

The matrix that determines the asymptotic behaviour of P_j is

$$\overline{P}_j = \lim_{k \to \infty} {P_j}^k = \overline{v}_j \overline{1}_j^T = (\overline{v}_j | \ldots | \overline{v}_j), \quad j = 1, \ldots, p.$$

This hypothesis (**H1**) reflects the modelization of a hierarchically organized system, because it answers the following properties:

(a) The unperturbed system represents the internal dynamics of every group $j = 1, \ldots, p$, i.e., the fast dynamics.
(b) The internal dynamics is conservative, that is, in absence of external interactions, the total number of individuals in every group remains constant.
(c) The internal dynamics has an asymptotically stable equilibrium distribution, represented in the group j by the vector $\overline{v}_j$, $j = 1, \ldots, p$.

We define now the new variables that we will call aggregated or global variables:

$$x_n{}^j = \sum_{k=1}^{N^j} x_n{}^{jk}, \quad j = 1, \ldots, p.$$

In order to obtain the linear system that will be satisfied by these new variables, we define the next matrices, whose properties are summarized in the following Lemma without proof:

$$\overline{P} = \text{diag}\left(\overline{P}_1, \ldots, \overline{P}_p\right); \quad \overline{P}_c = \text{diag}\left(\overline{v}_1, \ldots, \overline{v}_p\right); \quad U = \text{diag}\left(\overline{1}_1{}^T, \ldots, \overline{1}_p{}^T\right).$$

Lemma 1. Matrices $\overline{P}$, $\overline{P}_c$ and U verifies the following identities:

(a) $\overline{P}A(0) = A(0)\overline{P} = \overline{P} = \overline{P}\overline{P}$
(b) $A(0)\overline{P}_c = \overline{P}\overline{P}_c = \overline{P}_c$

(c) $U\overline{P} = U$; $U\overline{P}_c = I_p$; $\overline{P}_c U = \overline{P}$.

By multiplying both members of system (2.1) by matrix U,

$$UX_{n+1}(\epsilon) = UA(\epsilon)X_n(\epsilon)$$

we get the aggregated variables in the left member but we fail to obtain an autonomous system. To avoid this difficulty we consider the following system of order p:

$$UX_{n+1}(\epsilon) = UA(\epsilon)\overline{P}X_n(\epsilon)$$

where we are implicitly supposing that internal dynamics in every group has reached its equilibrium distribution, determined by matrix $\overline{P}$.

This new system is autonomous. Since $\overline{P} = \overline{P}_c U$, we have

$$UX_{n+1}(\epsilon) = UA(\epsilon)\overline{P}_c UX_n(\epsilon)$$

and denoting the new variables $Y_n(\epsilon) = UX_n(\epsilon)$, we get

$$Y_{n+1}(\epsilon) = B(\epsilon)Y_n(\epsilon) \tag{2.2}$$

where $B(\epsilon) = UA(\epsilon)\overline{P}_c$. Matrices $B(\epsilon)$ depend holomorphically on ϵ and $B(0) = I_p$.

The main object of this work is to get a comparison of the asymptotic behaviours of both systems, the general system (2.1) and the aggregated system (2.2). To reach this, it is enough to compare the dominant elements, eigenvalues and eigenvectors, of matrices $A(\epsilon)$ and $B(\epsilon)$.

We will use in our task the general theory of analytical perturbation of matrices. We summarize the main results that we will use in the Appendix, following [8] and [3].

Lemma 2. Let $A(0)$ be the unperturbed matrix of the general system (2.1). Then we have

(a) The strictly dominant eigenvalue of $A(0)$ is 1 and its algebraic multiplicity is p.
(b) 1 is a semisimple eigenvalue of $A(0)$, and a base of its eigenspace is

$$(\overline{\nu}_1|0|\dots|0)^T, \quad (0|\overline{\nu}_2|\dots|0)^T, \dots, \quad (0|0|\dots|\overline{\nu}_p)^T.$$

(c) The eigenprojection matrix of the eigenvalue 1 is $\overline{P}$.

In the aggregated system $B(0) = I_p$, and to compare the dominant elements of $A(\epsilon)$ and $B(\epsilon)$, we should compare the 1-groups of both. For this, we need to know the structure of the eigenvalues and eigenvectors of matrices $\overline{P}A'(0)\overline{P}$ and $B'(0)$ (see the Appendix).

Lemma 3. Let $\overline{P}(\mathbf{R}^N)$ be the eigenspace associated to the eigenvalue 1 of the matrix $A(0)$ and $\tilde{A}$ the restriction of the operator $\overline{P}A'(0)\overline{P}$ to this subspace. Then

(a) $\det\left(\tilde{A} - \lambda I\right) = \det\left(B'(0) - \lambda I\right)$.

(b) If $\overline{v} \in \overline{P}(\mathbf{R}^N)$ is an eigenvector associated to the eigenvalue λ, then $U\overline{v}$ is an eigenvector of $B'(0)$ associated to same eigenvalue λ.

(c) If $\overline{w} \in \mathbf{R}^p$ is an eigenvector of $B'(0)$ associated to the eigenvalue λ, then $\overline{P}_c\overline{w}$ is an eigenvector of $\overline{P}A'(0)\overline{P}$ associated to the same eigenvalue λ.

Let us notice that it is verified the next direct sum decomposition

$$\mathbf{R}^N = \overline{P}(\mathbf{R}^N) \oplus (I - \overline{P})(\mathbf{R}^N)$$

and that both spaces are invariant under the operator $\overline{P}A'(0)\overline{P}$; moreover, this operator vanishes identically on the subspace $(I - \overline{P})(\mathbf{R}^N)$. All that means that the characteristic polynomial of $\overline{P}A'(0)\overline{P}$ verifies

$$\det\left(\overline{P}A'(0)\overline{P} - \lambda I\right) = \lambda^{N-p}\det\left(B'(0) - \lambda I\right) = \lambda^{N-p}(\lambda - \lambda_1)^{m_1}\dots(\lambda - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r$ are the eigenvalues of $B'(0)$ with multiplicities $m_1, \dots, m_r$.

As $B(0) = I_p$, the 1-group of $B(\epsilon)$ is formed by all its eigenvalues. We could now state in the following theorem the main result we were looking for, and whose proof is a direct consequence of what we have just noticed.

Theorem 1. Let $A(\epsilon)$ be the matrix of the general system (2.1) and let $\lambda(\epsilon)$ represent the eigenvalues of the 1-group. Let $B(\epsilon)$ be the matrix of the aggregated system (2.2) and let $\mu(\epsilon)$ represent its eigenvalues. There exists $\delta > 0$ such that if $0 < |\epsilon| < \delta$, then the eigenvalues $\lambda(\epsilon)$ and $\mu(\epsilon)$ are classified in the $(1+\epsilon\lambda_s)$-groups, $s = 1, \dots, r$, verifying

(a) $\lambda(\epsilon)$ belongs to the $(1+\epsilon\lambda_s)$-group if and only if

$$\lambda(\epsilon) = 1 + \epsilon\lambda_s + O(\epsilon^{1+(1/p_s)}), \quad (\epsilon \to 0)$$

where p_s is an integer that satisfies $1 \le p_s \le m_s$.

(b) $\mu(\epsilon)$ belongs to the $(1+\epsilon\lambda_s)$-group if and only if

$$\mu(\epsilon) = 1 + \epsilon\lambda_s + O(\epsilon^{1+(1/q_s)}), \quad (\epsilon \to 0)$$

where q_s is a integer that satisfies $1 \le q_s \le m_s$. The dimension of the $(1+\epsilon\lambda_s)$-group is m_s in both cases.

(c) If $\overline{v}(\epsilon)$ is an eigenvector of $A(\epsilon)$ associated to the eigenvalue $\lambda(\epsilon)$ of the $(1+\epsilon\lambda_s)$-group, continuous at $\epsilon = 0$ and such that $\overline{v}(0) \neq 0$, then

$$\overline{P}A'(0)\overline{P}\overline{v}(0) = \lambda_s\overline{v}(0); \quad B'(0)\left(U\overline{v}(0)\right) = \lambda_s\left(U\overline{v}(0)\right).$$

(d) If $\overline{w}(\epsilon)$ is an eigenvector of $B(\epsilon)$ associated to the eigenvalue $\mu(\epsilon)$ of the $(1+\epsilon\lambda_s)$-group continuous at $\epsilon = 0$ and such that $\overline{w}(0) \neq 0$, then

$$B'(0)\overline{w}(0) = \lambda_s\overline{w}(0); \quad \overline{P}A'(0)\overline{P}\left(\overline{P}_c\overline{w}(0)\right) = \lambda_s\left(\overline{P}_c\overline{w}(0)\right).$$

Finally we make an additional hypothesis in order to get the systems (2.1) and (2.2) having an asymptotic behaviour governed by a strictly dominant eigenvalue.

Hypothesis (H2). Matrix $\overline{P}A'(0)\overline{P}$ has a simple nonzero eigenvalue μ whose real part is strictly greater than the real parts of the rest of the eigenvalues.

Let $\overline{v} = (v_1, \ldots, v_N)^T$ be an eigenvector of $\overline{P}A'(0)\overline{P}$ associated to μ.

Theorem 2. If $\epsilon \geq 0$, $|\epsilon| < \delta$, then matrix $A(\epsilon)$ has a simple eigenvalue $\lambda_{\max}(\epsilon)$ modulus strictly dominant that admits the following expansion

$$\lambda_{\max}(\epsilon) = 1 + \epsilon\mu + \epsilon f(\epsilon); \quad \lim_{\epsilon \to 0^+} f(\epsilon) = 0\,.$$

Associated to this eigenvalue there exists a unique eigenvector of $A(\epsilon)$ that could be written in the following way

$$\overline{x}(\epsilon) = \overline{v} + O(\epsilon), \quad (\epsilon \to 0)\,.$$

Moreover, $B(\epsilon)$ has a simple eigenvalue $\mu_{\max}(\epsilon)$ modulus strictly dominant that admits the following expansion

$$\mu_{\max}(\epsilon) = 1 + \epsilon\mu + \epsilon g(\epsilon); \quad \lim_{\epsilon \to 0^+} g(\epsilon) = 0$$

and an associated eigenvector

$$\overline{y}(\epsilon) = U\overline{v} + O(\epsilon), \quad (\epsilon \to 0)\,.$$

So, the asymptotic behaviour of system (2.1) is

$$\lim_{n \to +\infty} \frac{X_n(\epsilon)}{(1 + \epsilon\mu + \epsilon f(\epsilon))^n} = C_0\,(\overline{v} + O(\epsilon))\,, \quad (\epsilon \to 0)$$

and, therefore, the aggregated variables behave as follows

$$\lim_{n \to +\infty} \frac{UX_n(\epsilon)}{(1 + \epsilon\mu + \epsilon f(\epsilon))^n} = C_0\,(U\overline{v} + O(\epsilon))\,, \quad (\epsilon \to 0)\,.$$

Being that the asymptotic behaviour of the aggregated system (2.2) is

$$\lim_{n \to +\infty} \frac{Y_n(\epsilon)}{(1 + \epsilon\mu + \epsilon g(\epsilon))^n} = C_0\,(U\overline{v} + O(\epsilon))\,, \quad (\epsilon \to 0)$$

we have the similarity of behaviours we were looking for.

3. Generalized Leslie Models with Individual Activities

We are going to apply the above general results to the particular case of an age-structured population where we will distinguish several activities in every age class. We will use the following notation:

$$x_n{}^{ji} = \text{ number of individuals of age class } j \text{ in activity } i \text{ at time } n.$$

$$1 \leq i \leq N^j, \quad 1 \leq j \leq p, \quad \sum_{j=1}^{p} N^j = N, \quad n = 0, 1, \ldots$$

$$\overline{x}_n{}^j = (x_n{}^{j1}, \ldots, x_n{}^{jN^j})^T; \quad X_n = (\overline{x}_n{}^1, \ldots, \overline{x}_n{}^p)^T; \quad x_n{}^j = \sum_{i=1}^{N^j} x_n{}^{ji}.$$

We suppose that the changes of activity in age class j are represented by a regular stochastic matrix P_j ($N^j \times N^j$), and we keep the notation of the previous section.

The transference coefficients are divided in two classes as in the classical Leslie model:

Fertility coefficients

$$\begin{aligned} F_{kl}{}^j = &\text{ transference coefficient from age class } j \text{ and activity } l \\ &\text{ to age class 1 and activity } k. \\ &k = 1, \ldots, N^1; \quad l = 1, \ldots, N^j; \quad 1 \leq j \leq p; \end{aligned}$$

Aging coefficients

$$\begin{aligned} S_{kl}{}^j = &\text{ transference coefficient from age class } j \text{ and activity } l \\ &\text{ to age class } j+1 \text{ and activity } k. \\ &k = 1, \ldots, N^{j+1}; \quad l = 1, \ldots, N^j; \quad 1 \leq j \leq p-1. \end{aligned}$$

These coefficients verify

$F_{kl}{}^j \geq 0$ and there is some nonzero fertility coefficient of the last age class.

$S_{kl}{}^j \geq 0$ for every j not all vanishing, and $\sum_{k=1}^{N^{j+1}} S_{kl}{}^j \leq 1$.

We define submatrices

$$F_j = \left[F_{kl}{}^j\right]_{N^1 \times N^j}, \quad j = 1, \ldots, p; \quad S_j = \left[S_{kl}{}^j\right]_{N^{j+1} \times N^j}, \quad j = 1, \ldots, p-1$$

and finally a generalized Leslie matrix

$$L = \begin{bmatrix} F_1 & F_2 & \ldots & F_{p-1} & F_p \\ S_1 & 0 & \ldots & 0 & 0 \\ 0 & S_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & S_{p-1} & 0 \end{bmatrix}$$

In our model we must reflect the two different time scales, those associated to P and L, respectively. For that, we chose as unit time the projection interval corresponding to P and approximate (generally, it is not possible to do that exactly) the effect of L over that time interval, which is much shorter than its own projection interval, by the following matrix

$$L(\epsilon) = \epsilon L + (1-\epsilon) I_N, \quad 0 < \epsilon < 1.$$

$L(\epsilon)$ verifies the following property:

If L has a dominant eigenvalue λ with an associated eigenvector $\overline{v}$, then $L(\epsilon)$ has $\epsilon\lambda + (1-\epsilon)$ as strictly dominant eigenvalue and $\overline{v}$ is also its associated eigenvector.

We could yield then, that dynamics associated to L and $L(\epsilon)$ have the same asymptotically stable distribution, but L has a much greater growth rate than $L(\epsilon)$ because $\epsilon\lambda + (1-\epsilon)$ is much closer to 1 than λ.

Finally we propose the next generalized Leslie model with individual activities,

$$X_{n+1}(\epsilon) = [L(\epsilon)P]\, X_n(\epsilon)$$

where $P = \operatorname{diag}(P_1, \ldots, P_p)$.

Following the general construction of the previous section, we associate to this system the next aggregated one

$$Y_{n+1}(\epsilon) = \left[UL(\epsilon)P\overline{P}_c\right] Y_n(\epsilon) = \overline{L}(\epsilon)Y_n(\epsilon)$$

where we follow the notation

$$\overline{L}(\epsilon) = UL(\epsilon)P\overline{P}_c = \epsilon UL\overline{P}_c + (1-\epsilon)I_p\,.$$

It is direct to prove that $\overline{L} = UL\overline{P}_c$ is a classical Leslie matrix of order p whose coefficients are

$$\begin{aligned} f_j &= \overline{1}_1{}^T F_j \overline{v}_j, \ (j = 1, \ldots, p) \ \ \text{(fertility coefficients)} \\ s_j &= \overline{1}_{j+1}{}^T S_j \overline{v}_j, \ (j = 1, \ldots, p-1) \ \ \text{(survival coefficients)} \end{aligned}$$

and it is in this way that the aggregated system takes account of individual activities in every age class.

$\overline{L}$ is an irreducible matrix because $f_p \neq 0$, and so, if we denote by $\overline{\lambda}$ and $\overline{v}$ its dominant eigenvalue and eigenvector of positive components, we have that

$$\lim_{n\to+\infty} \frac{Y_n(\epsilon)}{\left(\epsilon\overline{\lambda} + (1-\epsilon)\right)^n} = C\overline{v}\,.$$

And for the general system we have the next asymptotic behaviour

$$\lim_{n\to+\infty} \frac{X_n(\epsilon)}{\left(\epsilon\overline{\lambda} + (1-\epsilon) + O(\epsilon^2)\right)^n} = C\left(\overline{P_c}\overline{v} + O(\epsilon)\right), \quad (\epsilon \to 0)$$

and so the similarity of behaviour between the aggregated system and the aggregated variables of the general system is clear.

4. Conclusion

In further work, we intend to describe the individual behaviour by use of the game theory ([4, 5]). For example, one could consider subpopulations corresponding to different strategies. The fast system would describe the game dynamics at the individual level. The slow system would describe the demography of the population.

In this work, the fast dynamics was linear. Interesting situations occur when the fast system is nonlinear. We shall investigate these nonlinear cases in our future contributions.

Appendix. General Results of the Theory of Analytical Perturbation of Matrices

Let X be a complex linear space of finite dimension N and let $T(\epsilon)$ be a linear operator defined on X, that admits the following expansion

$$T(\epsilon) = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots \quad , \quad |\epsilon| < R.$$

Let λ_0 be an eigenvalue of T_0 (unperturbed operator) and P_0 its associated eigenprojection with $\dim(P_0) = n \leq N$. The dependence of the eigenvalues $\lambda(\epsilon)$ of $T(\epsilon)$ on ϵ is continuous and it is called λ_0-group the subset of eigenvalues of $T(\epsilon)$ that verify $\lambda(\epsilon) \to \lambda_0$ when $\epsilon \to 0$.

The spectrum of operator $\tilde{T}_1 = P_0 T_1 P_0$ allows us to make a partition of the λ_0-group. Without loss of generality, we could suppose that all the eigenvalues of $\tilde{T}_1$ on $P_0 X$ are different from zero and as this operator vanishes identically on $(I - P_0)X$, we could write

$$\det(\tilde{T}_1 - \mu I) = \mu^{N-n}(\mu - \mu_1)^{m_1} \ldots (\mu - \mu_r)^{m_r}$$

with $\mu_i \neq \mu_j$ if $i \neq j$, $\mu_j \neq 0$, $j = 1, \ldots, r$, $m_1 + \cdots + m_r = n$.

The λ_0-group admits a partition in r subgroups corresponding to the r different eigenvalues $\mu_1, \ldots, \mu_r$, called $(\lambda_0 + \epsilon\mu_j)$-groups, $j = 1, \ldots, r$.

Theorem 3. Let $T(\epsilon)$ be a holomorphic perturbation of $T(0) = T_0$ and let λ_0 be a semisimple unperturbed eigenvalue, it exists $\delta > 0$ such that if $|\epsilon| < \delta$ and $\lambda(\epsilon)$ is a perturbed eigenvalue of the λ_0-group of $T(\epsilon)$, we have that $\lambda(\epsilon)$ belongs to the $(\lambda_0 + \epsilon\mu_j)$-group if and only if

$$\lambda(\epsilon) = \lambda_0 + \epsilon\mu_j + O(\epsilon^{1+(1/p)})$$

where p is an integer verifying $1 \leq p \leq m_j$.

Also, if $\overline{x}(\epsilon)$ is a perturbed eigenvector associated to the eigenvalue $\lambda(\epsilon)$ of the $(\lambda_0 + \epsilon\mu_j)$-group, continuous at $\epsilon = 0$ and with $\overline{x}(0) = \overline{x}_0 \neq 0$, then it is verified that

$$\tilde{T}_1 \overline{x}_0 = \mu_j \overline{x}_0 .$$

It is simple to prove the following result about dominant eigenvalues.

Lemma 4. Let λ_0 be a semisimple unperturbed eigenvalue of T_0 and modulus strictly dominant. Suppose that μ_1 is the eigenvalue of $\tilde{T}_1$ with the largest real part. Then the eigenvalues of the $(\lambda + \epsilon\mu_1)$-group are modulus strictly dominant for $T(\epsilon)$ when $\epsilon \geq 0$, $\epsilon \to 0$.

In particular, if μ_1 is simple, the eigenvalue of $T(\epsilon)$ that admits the following expansion

$$\lambda(\epsilon) = \lambda_0 + \epsilon\mu_1 + O(\epsilon^2); \quad (\epsilon \to 0)$$

is modulus strictly dominant when $\epsilon \geq 0$, $\epsilon \to 0$.

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