ON THE VALIRON DEFICIENCIES OF ORIENTED FUNCTIONS

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Abstract.

An extension of a result of D. F. Shea on meromorphic functions of order λ, $0 \leqq \lambda \leqq 1$ with negative zeros and positive poles is presented for the class of oriented functions introduced by G. Valiron. Let $\Delta(0, f)$, $\Delta(\infty, f)$ denote Valiron deficiencies and set $X = 1 - \Delta(0, f)$, $Y = 1 - \Delta(\infty, f)$. D. F. Shea proved for functions of order λ less than one with negative zeros and positive poles that $X^2 + Y^2 - 2XY \cdot \cos \pi\lambda \leqq \sin^2 \pi\lambda$. In this paper it is proved that this relationship still holds for oriented functions with sets of zeros and poles symmetric with respect to the real axis. Furthermore if $X = Y$, in particular if the zeros are symmetric to the poles with respect o the imaginary axis, then $X^2 \leqq 2^{-1}(1 + \cos \pi\lambda)$ for functions not necessarily oriented but with zeros and poles $a_\nu = |a_\nu| e^{i\theta_\nu}$, $b_\nu = |b_\nu| e^{i\gamma_\nu}$ satisfying $\pi - |\theta_\nu| \leqq \beta_1 < \pi/2$, $|\gamma_\nu| \leqq \beta_1 < \pi/2$ and being more densely distributed around the real axis than away from it.

1. Introduction.

In [7] D. F. Shea proved the following result

THEOREM A. *Let $f(z)$ be a meromorphic function in the plane of order λ, $0 \leqq \lambda \leqq 1$, whose zeros lie on the negative real axis and whose poles lie on the positive real axis.*

If we set

$X = 1 - \Delta(0, f)$, $Y = 1 - \Delta(\infty, f)$,

where $\Delta(0, f)$ denote Valiron deficiencies, then it holds

$$X^2 + Y^2 - 2XY \cos \pi\lambda \leqq \sin^2 \pi\lambda \tag{1.1}$$

when $1/2 \leqq \lambda \leqq 1$.

If $\lambda < 1/2$, (1.1) *also holds provided*

$$X \geqq \cos \pi\lambda \text{ and } Y \geqq \cos \pi\lambda.$$

He also suggests that this and other related results of [7], might be extended to functions with more general distributions of zeros and poles.

In particular he thinks that such an extension is possible for the class of

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oriented functions introduced by Valiron [9]. Extending the terminology of Valiron to meromorphic functions, the oriented functions are defined as those meromorphic functions whose zeros $\{a_\nu\}$ and poles $\{b_\nu\}$ satisfy

$$\lim_{\nu\to\infty} |\operatorname{Arg} a_\nu| = \pi, \quad \lim_{\nu\to\infty} \operatorname{Arg} b_\nu = 0.$$

In this paper we intend to prove such an extension for Theorem A. However we must impose a condition of symmetry of zeros and poles with respect to the real axis.

Our theorem will be in particular applicable to the functions

$$G_\lambda(z) = F_\lambda^1(z)\, F_\lambda^2(z),$$

where $F_\lambda^1(z)\, F_\lambda^2(z)$ are the "modified Lindelöf functions"

$$F_\lambda^1(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{\nu^\alpha e^{i\theta_\nu}}\right), \quad F_\lambda^2(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{\nu^\alpha e^{-i\theta_\nu}}\right)$$

$$\alpha = 1/\lambda,\ 0 < \lambda < 1,\ \sum_{\nu=1}^{\infty} |\theta_\nu| < \infty.$$

Furthermore, if there is also symmetry between zeros and poles, the conclusions of Theorem A still hold for functions not necessarily oriented but with zeros and poles whose arguments satisfy $\pi - |\theta_\nu| \leqq \beta_1$, $|\gamma_\nu| \leqq \beta_1$, $0 \leqq \beta_1 < \pi/2$.

2. Statements of the results.

We shall deduce the above mentioned results from the following one.

THEOREM 1. *Let $f(z)$ be a meromorphic function in the plane of order λ and lower order μ, with $0 < \mu \leqq \lambda < 1$, and whose zeros and poles $a_\nu = |a_\nu|e^{i\theta_\nu}$, $b_\nu = |b_\nu|e^{i\gamma_\nu}$ satisfy the following conditions*

(2.1) $$\{a_\nu\}_{\nu\in N} = \{\bar{a}_\nu\}_{\nu\in N},\ \{b_\nu\}_{\nu\in N} = \{\bar{b}_\nu\}_{\nu\in N},$$

(2.2) $$0 \leqq \pi - |\theta_\nu| \leqq \beta_1 < \pi/2,\ 0 \leqq |\gamma_\nu| \leqq \beta_1 < \pi/2,$$

and

(2.3) $$\sum_{|a_\nu|\leqq r} \pi - |\theta_\nu| = O(r^{\mu'}),\ \sum_{|a_\nu|\leqq r} = O(r^{\mu'}),\ \mu' < \mu.$$

Then for all β, $\beta_1 < \beta < \pi - \beta_1$, we have the relationship

(2.4) $$\sin \pi\lambda \geqq X \sin \beta\lambda + Y \sin(\pi - \beta)\lambda.$$

As consequences of Theorem 1 we shall obtain

THEOREM 2. *Let $f(z)$ a be meromorphic function of order λ and lower order μ, with $0 < \mu \leqq \lambda < 1$, and whose zeros and poles $a_\nu = |a_\nu| e^{i\theta_\nu}$, $b_\nu = |b_\nu| e^{i\gamma_\nu}$ satisfy* (2.1) *and* (2.3) *of Theorem* 1 *and*

$$\lim_{\nu \to \infty} |\theta_\nu| = \pi, \quad \lim_{\nu \to \infty} \gamma_\nu = 0. \tag{2.5}$$

Then we have the relationship (1.1) *of Theorem* A, i.e.

$$X^2 + Y^2 - 2XY \cos \pi\lambda \leqq \sin^2 \pi\lambda \tag{2.6}$$

when $1/2 \leqq \lambda < 1$.

If $\lambda < 1/2$, (2.6) *still holds provided*

$$X \geqq \cos \pi\lambda \text{ and } Y \geqq \cos \pi\lambda. \tag{2.7}$$

THEOREM 3. *Let $f(z)$ be a meromorphic function of order λ and lower order μ, with $0 < \mu \leqq \lambda < 1$, and whose zeros and poles $a_\nu = |a_\nu| e^{i\theta_\nu}$, $b_\nu = |b_\nu| e^{i\gamma_\nu}$ satisfy* (2.1), (2.2) *and* (2.3) *of Theorem* 1. *Then if $X = Y$ we have*

$$X^2 \leqq \frac{1 + \cos \pi\lambda}{2}. \tag{2.8}$$

The condition $X = Y$ is fulfilled in particular in the case of symmetry of zeros and poles with respect to the imaginary axis and more general if $|a_\nu| = |b_\nu|$.

3. Auxiliary results.

We present some integral representations of finite genus q. We present them in the general case though our main interest will be $q = 0$.

Let

$$E(u, q) = (1 - u) \exp\left(u + \frac{u^2}{2} + \ldots + \frac{u^q}{q}\right), \; (q > 0)$$

$$E(u, 0) = 1 - u,$$

then we have the following formula due to Valiron

$$\log E\left(-\frac{z}{a}, q\right) = (-1)^q \int_{L_a} \frac{z^{q+1}}{t^{q+1}(t + z)} \, dt, \tag{3.1}$$

valid in $|\arg z - \alpha| < \pi$, where $\alpha = \operatorname{Arg} a$, $a \neq 0$ and

$$L_a = \{te^{i\alpha}, \; |a| \leqq t \, \infty\}.$$

Formula (3.1) can easily be obtained by expressing the integrand as a power series and then by a term-by-term integration.

From (3.1), it follows when $\alpha \neq \pi$

$$\int_0^\beta \log\left|\left(-\frac{re^{i\theta}}{a}, q\right)\right| d\theta \tag{3.2}$$
$$= (-1)^{q+1}\,\mathrm{Re}\left\{i\int_{\Gamma_r} z^q\,dz \int_{L_a} \frac{dt}{t^{q+1}(t+z)}\right\},$$

where $\Gamma_r = \{re^{i\theta}\,|\,0 \leqq \theta \leqq \beta, 0 < \beta < \pi - |\alpha|\}$.

Let now $g(z)$ be the Weierstrass product of genus q

$$g(z) = \prod_{\nu=1}^{\infty} E\left(-\frac{z}{a_\nu}, q\right),\ a_\nu \neq 0,$$

we have the following result.

LEMMA 1. *If the zeros* $a_\nu = |a_\nu|e^{i\theta_\nu}$ *satisfy*

$$|\theta_\nu| \leqq \beta_1 < \pi/2, \tag{3.3}$$

$$\sum_{|a_\nu| \leqq r} |\theta_\nu| = O(r^{\mu'}),\ 0 < \mu' < \mu,\ \mu' < q + 1 \tag{3.4}$$

then we have

$$\int_0^\beta \log|g(re^{i\theta})|\,d\theta \tag{3.5}$$
$$= (-1)^{q+1}\,\mathrm{Re}\left\{i\int_0^\infty \frac{n(t,0,g)}{t^{q+1}}\,dt \int_{\Gamma_r} \frac{z^q\,dz}{t+z}\right\} + O(r^{q+\mu'}),$$

where $n(t,0,g)$ *is the counting function of the zeros of* g *and*

$$\Gamma_r = \{re^{i\theta}\,|\,0 \leqq \theta \leqq \beta < \pi - \beta_1\}.$$

PROOF OF LEMMA 1.

$$\int_0^\beta \log|g(re^{i\theta})|\,d\theta = \int_0^\beta \log\left|\prod_{\nu=1}^{\infty} E\left(-\frac{re^{i\theta}}{a_\nu}, q\right)\right| d\theta \tag{3.6}$$
$$= \sum_{\nu=1}^{\infty} \int_0^\beta \log\left|E\left(-\frac{re^{i\theta}}{a_\nu}, q\right)\right| d\theta.$$

Making use of (3.2) and choosing β as in the statement of the lemma we deduce from (3.6)

$$\int_0^\beta \log|g(re^{i\theta})|\,d\theta \tag{3.7}$$
$$= (-1)^{q+1}\,\mathrm{Re}\left\{i\sum_{\nu=1}^{\infty} \int_{L_{a_\nu}} \frac{dt}{t^{q+1}} \int_{\Gamma_r} \frac{z^q dz}{t+z}\right\}.$$

Now we have

$$
\begin{aligned}
(3.8)\qquad & \left|(-1)^{q+1}\operatorname{Re}\left\{i\sum_{\nu=1}^{\infty}\int_{L_{a_\nu}}\frac{dt}{t^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{t+z}\right\}\right.\\
&\left.-(-1)^{q+1}\operatorname{Re}\left\{i\int_0^{\infty}\frac{n(t,0,g)}{t^{q+1}}\,dt\int_{\Gamma_r}\frac{z^q\,dz}{t+z}\right\}\right|\\
&=\left|(-1)^{q+1}\operatorname{Re}\left\{i\sum_{\nu=1}^{\infty}\int_{L_{a_\nu}}\frac{dt}{t^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{t+z}\right\}\right.\\
&\left.-(-1)^{q+1}\operatorname{Re}\left\{i\int_{|a_\nu|}^{\infty}\frac{dt}{t^{q+1}}t\int_{\Gamma_r}\frac{z^q\,dz}{t+z}\right\}\right|
\end{aligned}
$$

We get the estimates

$$
\begin{aligned}
(3.9)\qquad & \left|\int_{L_{a_\nu}}\frac{dt}{t^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{t+z}-\int_{|a_\nu|}^{\infty}\frac{ds}{s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{s+z}\right|\\
&\leqq\left|\int_{|a_\nu|}^{\infty}\frac{e^{i\theta_\nu}\,ds}{e^{i(q+1)\theta_\nu}s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{se^{i\theta_\nu}+z}\right.\\
&\left.-\int_{|a_\nu|}^{\infty}\frac{ds}{s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{se^{i\theta_\nu}+z}\right|\\
&+\left|\int_{|a_\nu|}^{\infty}\frac{ds}{s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{se^{i\theta_\nu}+z}-\int_{|a_\nu|}^{\infty}\frac{ds}{s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{s+z}\right|\\
&=I_\nu+II_\nu.
\end{aligned}
$$

First we estimate I_ν

$$
\begin{aligned}
(3.10)\qquad I_\nu &=\left|(1-e^{-iq\theta_\nu})\int_{|a_\nu|}^{\infty}\frac{ds}{s^{q+1}}\int_{\Gamma_r}\frac{z^q\,dz}{se^{i\theta_\nu}+z}\right|\\
&\leqq\alpha_\nu\int_{|a_\nu|}^{\infty}\frac{\pi r^{q+1}}{\sqrt{r^2+s^2-2rs\eta}}\,\frac{ds}{s^{q+1}}
\end{aligned}
$$

where

$$\alpha_\nu=|1-e^{-iq\theta_\nu}|=\sqrt{2(1-\cos q\theta_\nu)}\leqq C|\theta_\nu|$$

and

$$\eta=\cos(\pi-\beta_1-\beta)<1$$

Making use of (3.4), we obtain from (3.10)

$$\sum_{\nu=1}^{\infty} I_\nu \leqq \sum_{\nu=1}^{\infty} \alpha_\nu \int_{|a_\nu|}^{\infty} \frac{\pi r^{q+1}}{\sqrt{r^2+s^2-2rs\eta}} \frac{ds}{s^{q+1}} \tag{3.11}$$

$$= \sum_{\nu=1}^{\infty} \int_{|a_\nu|}^{|a_{\nu+1}|} \left(\sum_{k \leqq \nu} \alpha_k\right) \frac{\pi r^{q+1}}{\sqrt{r^2+s^2-2rs\eta}} \frac{ds}{s^{q+1}}$$

$$\leqq C \int_{\delta=\min_{\nu\in N}|a_\nu|}^{\infty} \frac{\pi r^{q+1}}{\sqrt{r^2+s^2-2rs\eta}} \frac{s^{\mu'}ds}{s^{q+1}}$$

$$= C r^{\mu'} \int_{\delta/r}^{\infty} \frac{\pi}{t^{q-\mu'+1}} \frac{dt}{\sqrt{1+t^2-2t\eta}},$$

after making the change of variable $t = s/r$.

From (3.11) and L'Hôpital we have for $q > 0$

$$\lim_{r\to\infty} \frac{\displaystyle\int_{\delta/r}^{\infty} \frac{\pi}{t^{q-\mu'+1}} \frac{dt}{\sqrt{1+t^2-2t\eta}}}{r^q} \tag{3.12}$$

$$= \lim_{r\to\infty} \frac{\left[-\dfrac{\pi}{(\delta/r)^{q-\mu'+1}} \dfrac{1}{\sqrt{1+(\delta/r)^2-2(\delta/r)\eta}}\right]\left(-\dfrac{\delta}{r^2}\right)}{qr^{q-1}}$$

$$= \lim_{r\to\infty} \frac{\pi r^{q-\mu'+1}}{\delta^{q-\mu'} q r^q \sqrt{r^2+\delta^2-2\delta r\eta}} < \infty.$$

In the case $q = 0$,

$$\int_0^{\infty} \frac{\pi}{t^{q-\mu'+1}} \frac{dt}{\sqrt{1+t^2-2t\eta}} < \infty.$$

Thus from (3.11) and (3.12) we obtain

$$\sum_{\nu=1}^{\infty} I_\nu = O(r^{\mu'+q}). \tag{3.13}$$

As for II_ν we have

$$II_\nu = \left|(1-e^{i\theta_\nu}) \int_{|a_\nu|}^{\infty} \frac{ds}{s^{q+1}} \int_{\Gamma_r} \frac{z^q s ds}{(se^{i\theta_\nu}+z)(s+z)}\right. \tag{3.14}$$

$$\leqq \lambda_\nu \int_{|a_\nu|}^{\infty} \frac{\pi r^{q+1} s}{r^2+s^2-2rs\eta} \frac{ds}{s^{q+1}},$$

where $\lambda_\nu = |1 - e^{i\theta_\nu}| = \sqrt{2(1 - \cos\theta_\nu)} \leqq C|\theta_\nu|$ and $\eta < 1$ is the same as above. Using (3.4) again we get from (3.14)

$$\sum_{\nu=1}^{\infty} II_\nu \leqq \sum_{\nu=1}^{\infty} \int_{|a_\nu|}^{|a_{\nu+1}|} \left(\sum_{k \leqq \nu} \lambda_\nu\right) \frac{\pi r^{q+1} s}{r^2 + s^2 - 2rs\eta} \frac{ds}{s^{q+1}} \tag{3.15}$$

$$\leqq C \int_\delta^\infty \frac{\pi r^{q+1} s}{r^2 + s^2 - 2rs\eta} \frac{s^{\mu'} ds}{s^{q+1}}$$

$$= C r^{\mu'} \int_{\delta/r}^\infty \frac{\pi}{t^{q-\mu'+1}} \frac{tdt}{1 + t^2 - 2t\eta}.$$

Applying L'Hôpital rule again we obtain for $q > 0$

$$\lim_{r\to\infty} \frac{\displaystyle\int_{\delta/r}^\infty \frac{\pi}{t^{q-\mu'+1}} \frac{tdt}{1 + t^2 - 2t\eta}}{r^q} \tag{3.16}$$

$$= \lim_{r\to\infty} \frac{\left[-\dfrac{\pi}{(\delta/r)^{q-\mu'+1}} \dfrac{\delta/r}{1 + (\delta/r)^2 - 2(\delta/r)\eta}\right]\left(-\dfrac{\delta}{r^2}\right)}{qr^{q-1}}$$

$$= \lim_{r\to\infty} \frac{\pi\delta^{1-q+\mu'} r^{q-\mu'+1}}{qr^q(r^2 + \delta^2 - 2\delta r\eta)} = 0.$$

Again in the case $q = 0$.

$$\int_0^\infty \frac{\pi}{t^{q-\mu'+1}} \cdot \frac{tdt}{1 + t^2 - 2t\eta} < \infty.$$

Thus from (3.15) and (3.16) we conclude

$$\sum_{\nu=1}^{\infty} II_\nu = O(r^{q+\mu'}). \tag{3.17}$$

Finally from (3.7), (3.8), (3.9), (3.13) and (3.17) we get (3.5) of Lemma 1.

We now refer to [7] to see how the term

$$\frac{(-1)^{q+1}}{\pi} \operatorname{Re}\left\{i \int_0^\infty \frac{n(t,0,g)}{t^{q+1}} dt \int_{\Gamma_r} \frac{z^q dz}{t + z}\right\}$$

can be transformed by integration by parts into

$$\int_0^\infty N(t,0,g) K_q(t,r,\beta) dt$$

where the kernel $K_q(t,r,\beta)$ is defined by

$$K_q(t,r,\beta) = \frac{(-1)^q}{\pi}\left(\frac{r}{t}\right)^{q+1} \frac{r\sin q\beta + t\sin(q+1)\beta}{t^2 + 2tr\cos\beta + r^2}, \qquad q \geqq 0, -\pi < \beta < \pi.$$

The kernel $K_0(t,r,\beta)$ reduces to the Poisson kernel for the upper half plane, however we shall keep the above notation.

The properties of K_q which we shall make use of in what follows are

$$K_q(t,r,\theta) \geqq 0, \quad 0 \leqq \theta \leqq \frac{\pi}{q+1} \tag{3.18}$$

$$\int_0^\infty s^\lambda K_q(s,1,\beta)ds = \frac{\sin\beta\lambda}{\sin\pi\lambda}, \ (q < \lambda < q+1, \ -\pi < \beta < \pi) \tag{3.19}$$

To end up with this section, we rewrite the conclusion (3.5) of Lemma 1 in terms of the kernel K_q

$$\int_0^\beta \log|g(re^{i\theta})|\,d\theta = \int_0^\infty N(t,0,g)K_q(t,r,\beta)\,dt + O(r^{q+\mu'}). \tag{3.20}$$

4. Proof of Theorem 1.

The steps of the proof of Theorem 1 are now quite the same to those of the proof of Theorem A in [7].

Let f be a meromorphic function as in the statement of Theorem 1. Then f can be expressed in the form

$$f(z) = cz^m \frac{\prod_{\nu=1}^{\infty}\left(1 - \dfrac{z}{a_\nu}\right)}{\prod_{\nu=1}^{\infty}\left(1 - \dfrac{z}{b_\nu}\right)}.$$

We consider the function $g(z) = c^{-1}z^{-m}f(z)$ and apply Lemma 1. We have

$$\begin{aligned}\frac{1}{\pi}\int_0^\beta &\log|g(re^{i\theta})|\,d\theta \\ &= \frac{1}{\pi}\int_0^\beta \log\left|\prod_{\nu=1}^{\infty}\left(1 - \frac{re^{i\theta}}{a_\nu}\right)\right| d\theta - \frac{1}{\pi}\int_0^\beta \log\left|\prod_{\nu=1}^{\infty}\left(1 - \frac{re^{i\theta}}{b_\nu}\right)\right| d\theta.\end{aligned} \tag{4.1}$$

By (3.20) we have

$$\frac{1}{\pi}\int_0^\beta \log\left|\prod_{\nu=1}^{\infty}\left(1 - \frac{re^{i\theta}}{a_\nu}\right)\right| d\theta = \int_0^\infty N(t,0,g)K_o(t,r,\beta)dt + O(r^{\mu'}) \tag{4.2}$$

for any β, $0 \leqq \beta < \pi - \beta_1$.

We also have by (3.20) and Jensen's formula

$$
\begin{aligned}
(4.3)\qquad & \frac{1}{\pi}\int_0^{\beta} \log\left|\prod_{\nu=1}^{\infty}\left(1-\frac{re^{i\theta}}{b_\nu}\right)\right| d\theta \\
&= \frac{1}{\pi}\int_0^{\beta} \log\left|\prod_{\nu=1}^{\infty}\left(1+\frac{re^{i(\pi-\theta)}}{\bar{b}_\nu}\right)\right| d\theta \\
&= \frac{1}{\pi}\int_{\pi-\beta}^{\pi} \log\left|\prod_{\nu=1}^{\infty}\left(1+\frac{re^{i\theta}}{\bar{b}_\nu}\right)\right| d\theta \\
&= N(r,\infty,g) - \int_0^{\pi-\beta} \log\left|\prod_{\nu=1}^{\infty}\left(1+\frac{re^{i\theta}}{\bar{b}_\nu}\right)\right| d\theta \\
&= N(r,\infty,g) - \int_0^{\infty} N(t,\infty,g)K_o(t,r,\pi-\beta)\,dt + O(r^{\mu'}),
\end{aligned}
$$

for those β such that $0 \leqq \pi - \beta < \pi - \beta_1$ or equivalently $0 < \beta_1 < \beta$.

We note that we have made use of the fact

$$
\left|\prod_{\nu=1}^{\infty}\left(1+\frac{re^{i\theta}}{\bar{b}_\nu}\right)\right| = \left|\prod_{\nu=1}^{\infty}\left(1+\frac{re^{-i\theta}}{\bar{b}_\nu}\right)\right|
$$

which holds by (2.1).

From (4.1, (4.2) and (4.3) we conclude

$$
\begin{aligned}
(4.4)\qquad & \frac{1}{\pi}\int_0^{\beta} \log|g(re^{i\theta})|\, d\theta + N(r,\infty,g) \\
&= \int_0^{\infty} N(t,0,g)K_o(t,r,\beta)\,\mathrm{d}t + \int_0^{\infty} N(t,\infty,g)K_o(t,r,\pi-\beta)dt + O(r^{\mu'}),
\end{aligned}
$$

for those β such that $\beta_1 < \beta < \pi - \beta_1$.

Making use of the fact that $|g(re^{i\theta})|$ is an even function of θ, i.e. $|g(re^{i\theta})| = |g(re^{-i\theta})|$ what is a consequence again of (2.1), we get from (4.4)

$$
(4.5)\qquad m(r,f) \geqq m(r,g) + C\log r \geqq \frac{1}{\pi}\int_0^{\beta} \log|g(re^{i\theta})|\, d\theta + C\log r
$$

and since

$$
N(t,0,f) = N(t,0,g) + C\log t,\; N(t,\infty,f) = N(t,\infty,g) + C\log t
$$

we conclude from (4.5)

$$(4.6)\qquad \begin{aligned} T(r,f) &= m(r,f) + N(r,\infty,f) \\ &\geqq \frac{1}{\pi}\int_0^{\beta} \log|g(re^{i\theta})|\,d\theta + N(r,\infty,g) + O(\log r) \\ &\geqq \int_0^{\infty} N(t,0,f)\,K_o(t,r,\beta)\,dt + \int_0^{\infty} N(t,\infty,f)K_o(t,r,\pi-\beta)dt + O(r^{\mu'}). \end{aligned}$$

If $X = 1 - \Delta(0,f)$, choose $\bar{X}$ so that $0 < \bar{X} < X$ and if $X = 0$, take $\bar{X} = 0$. Hence, we always have

$$(4.7)\qquad N(t,0,f) \geqq \bar{X}T(t,f)$$

for all sufficiently large t. We take $\bar{Y}$ in a similar way, i.e.

$$1 - \Delta(\infty,f) = Y > \bar{Y} > 0$$

if $Y > 0$, and $\bar{Y} = 0$ if $Y = 0$, so that we also have

$$(4.8)\qquad N(t,\infty,f) \geqq \bar{Y}T(t,f).$$

Let $\{r_m\}$ be a sequence of Pŏlya peaks of the second kind of order λ for the function $T(t,f)$, see [7]). Then we obtain from (4.6), (4.7) and (4.8), bearing in mind that $K_o(t,r,\beta)$ is positive

$$\begin{aligned} T(r_m) &\geqq \bar{X}T(r_m)(1+o(1))\int_{s_m}^{S_m} (t/r_m^{\lambda})\,K_o(t,r_m,\beta)\,dt \\ &\quad + \bar{Y}T(r_m)(1+o(1))\int_{s_m}^{S_m} (t/r_m^{\lambda})\,K_o(t,r_m,\pi-\beta)\,dt + O(r_m^{\mu'}) \end{aligned}$$

as $m \to \infty$.

Making the change of variable $s = t/r_m$, dividing by $T(r_m)$ and evaluating the resulting integral according to (3.19) we get to

$$\sin\pi\lambda \geqq \bar{X}\sin\beta\lambda + \bar{Y}\sin(\pi-\beta)\lambda,$$

and finally letting $\bar{X} \to X$, $\bar{Y} \to Y$ we obtain

$$\sin\pi\lambda \geqq X\sin\beta\lambda + Y\sin(\pi-\beta)\lambda,$$

i.e. we conclude (2.4) of Theorem 1.

5. Proofs of Theorems 2 and 3.

PROOF OF THEOREM 2. We derive Theorem 2 from Theorem 1. We decompose f as

$$(5.1)\qquad f(z) = f^N(z)f_N(z)$$

where

$$f^N(z) = cz^m \prod_{\nu=1}^{N} \frac{\left(1 - \dfrac{z}{a_\nu}\right)}{\left(1 - \dfrac{z}{b_\nu}\right)}$$

and

$$f_N(z) = \prod_{\nu>N} \frac{\left(1 - \dfrac{z}{a_\nu}\right)}{\left(1 - \dfrac{z}{b_\nu}\right)}$$

Since f^N is rational, we deduce from (5.1)

$$T(r,f) = T(r,f_N) + O(\log r),$$

$$N(r,\infty,f) = N(r,\infty,f_N) + O(\log r),$$

$$N(r,0,f) = N(r,0,f_N) + O(\log r),$$

whence, writing $X_N = 1 - \Delta(0,f_N)$, $Y_N = 1 - \Delta(\infty,f_N)$, we have

$$X_N = X,\ Y_N = Y. \tag{5.2}$$

Making use of the hypothese (2.5), we deduce that given any $\beta_1, 0 < \beta_1 < \pi/2$, we can find N so large that

$$|\gamma_\nu| < \beta_1, |\pi - \theta_\nu| < \beta_1,\ \nu > N$$

so that we can apply Theorem 1 to f_N and conclude the relationship (2.4) for any $\beta, 0 < \beta < \pi$, and by continuity also for $\beta = 0, \pi$. Taking now

$$\beta = \frac{1}{\lambda}\operatorname{arctg}\left(\frac{X - Y\cos\pi\lambda}{Y\sin\pi\lambda}\right)$$

one gets (2.6) from (2.4) after some simple calculations.

Proof of Theorem 3. Theorem 3 is an immediate consequence of Theorem 1 since the inequality (2.8) is (2.4) for the particular case $X = Y$, $\beta = \pi/2$.

6. Example.

We show that the functions

$$G_\lambda(z) = F^1_\lambda(z)F^2_\lambda(z)$$

where $F^1_\lambda(z)$, $F^2_\lambda(z)$ are the modified Lindelöf functions

$$F_\lambda^1(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{\nu^\alpha e^{i\theta_\nu}}\right), F_\lambda^2(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{\nu^\alpha e^{-i\theta}\nu}\right)$$

$$\alpha = 1/\lambda, 0 < \lambda < 1, \sum_{\nu=1}^{\infty} |\theta_\nu| < \infty.$$

satisfy the hypotheses of Theorem 2.

We just check that $G_\gamma(z)$ is an integral function of regular growth of order λ.

We compare the asymptotic behaviour of a modified Lindelöf function, say $F_\lambda^1(z)$, with that of the corresponding Lindelöf function

$$F_\lambda(z) = \prod_{\nu=1}^{\infty}\left(1 + \frac{z}{\nu^\alpha}\right), \quad \alpha = 1/\lambda, 0 < \lambda < 1.$$

First we estimate $|F_\lambda^1(z)/F_\lambda(z)|$ in $|\mathrm{Arg}\, z| < \pi/2 - \delta, \delta > 0$.

$$\left|\frac{F_\lambda^1(z)}{F_\lambda(z)}\right| = \prod_{\nu=1}^{\infty}\left|\frac{1 + \dfrac{z}{\nu^\alpha e^{i\theta_\nu}}}{1 + \dfrac{z}{\nu^\alpha}}\right| \tag{6.1}$$

$$\leqq \prod_{\nu=1}^{\infty}\left[1 + \frac{|z|\,|1 - e^{i\theta_\nu}|}{|\nu^\alpha + z|}\right]$$

$$\leqq \exp\left\{\sum_{\nu=1}^{\infty}\frac{|z|\,|1 - e^{i\theta_\nu}|}{|\nu^\alpha + z|}\right\}.$$

The terms $|z|\,|\nu^\alpha + z|^{-1}$ are bounded by one in $|\mathrm{Arg}\, z| \leqq \pi/2 - \delta$. Since $|1 - e^{i\theta_\nu}| \leqq C\,|\theta_\nu|$, we conclude from (6.1)

$$\left|\frac{F_\lambda^1(z)}{F_\lambda(z)}\right| \leqq C < \infty \tag{6.2}$$

in $|\mathrm{Arg}\, z| \leqq \pi/2 - \delta$.

Similarly we have

$$\left|\frac{F_\lambda(z)}{F_\lambda^1(z)}\right| = \left|\frac{F_\lambda^N(z)}{F_\lambda^{1N}(z)}\right| \prod_{\nu > N}\left|\frac{1 + \dfrac{z}{\nu^\alpha}}{1 + \dfrac{z}{\nu^\alpha e^{i\theta_\nu}}}\right| \tag{6.3}$$

where

$$\prod_{\nu > N} \left| \frac{1 + \dfrac{Z}{\nu^\alpha}}{1 + \dfrac{z}{\nu^\alpha e^{i\theta_\nu}}} \right| \leqq \prod_{\nu > N} \left[1 + \frac{|z|\,|e^{i\theta_\nu} - 1|}{|\nu^\alpha e^{i\theta_\nu} + z|} \right] \tag{6.4}$$

$$\leqq \exp \left\{ \sum_{\nu > N} \frac{|z|\,|e^{i\theta_\nu} - 1|}{|\nu^\alpha e^{i\theta_\nu} + z|} \right\}.$$

We now take N so large that we have $|\theta_\nu| \leqq \delta$ for $\nu > N$. Thus we obtain again $|z|\,|\nu^\alpha e^{i\theta_\nu} + z|^{-1} \leqq 1$ for z in $|\operatorname{Arg} z| \leqq \pi - \delta$ and $\nu > N$.

Thus we deduce from (6.3) and (6.4)

$$\left| \frac{F_\lambda(z)}{F_\lambda^1(z)} \right| \leqq C \left| \frac{F_\lambda^N(z)}{F_\lambda^{1N}(z)} \right| \tag{6.5}$$

We have (see R. Nevanlinna [5], page 18)

$$\log |F_\lambda(z)| = (1 + \varepsilon(z)) \frac{\pi}{\sin \pi\lambda} |z|^\lambda \cos(\lambda \operatorname{Arg} z), \tag{6.6}$$

where $\varepsilon(z) \to 0$ uniformly in $|\operatorname{Arg} z| \leqq \pi - \delta$ as $|z| \to \infty$.

Bearing in mind

$$|\log |F_\lambda^N(re^{i\theta})|| = O(\log r), \quad |\log| F_\lambda^{1N}(re^{i\theta})|| = O(\log r),$$

we conclude from (6.2), (6.5) and (6.6)

$$C_1 r^\lambda \leqq \log^+ |F_\lambda^1(re^{i\theta})| = \log |F_\lambda^1(re^{i\theta})| \leqq C_2 r^\lambda,\ C_1, C_2 > 0,$$

in $|\operatorname{Arg} z| \leqq \pi/2 - \delta$.

Thus for $G_\lambda(z) = F_\lambda^1(z) F_\lambda^2(z)$ we have a relationship of the same kind, i.e.

$$C_1 r^\lambda \leqq \log^+ |G_\lambda(re^{i\theta})| \leqq C_2 r^\lambda, \quad C_1, C_2 > 0. \tag{6.7}$$

It can easily be checked that

$$\log^+ |G_\lambda(re^{i\theta})| = O(r^\lambda) \text{ in } \pi \geqq |\operatorname{Arg} z| \geqq \pi/2 - \delta,$$

and thus

$$(6.8)\quad T(r, G_\lambda) = m(r, G_\lambda) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |G_\lambda(re^{i\theta})|\, d\theta$$

$$= \frac{1}{2\pi} \int_{|\theta| \leqq \pi/2-\delta} \log^+ |G_\lambda(re^{i\theta})|\, d\theta + \frac{1}{2\pi} \int_{\pi/2-\delta \leqq |\theta| \leqq \pi} \log^+ |G_\lambda(re^{i\theta})|\, d\theta$$

$$= O(r^\lambda) + \frac{1}{2\pi} \int_{|\theta| \leqq \pi/2-\delta} \log^+ |G_\lambda(re^{i\theta})|\, d\theta.$$

Finally from (6.7) and (6.8) we conclude that $G_\lambda(z)$ is of order λ and regular growth.

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