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LINEAR OPERATORS AND VECTOR INTEGRALS

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Abstract. The object of this paper is to state several theorems of representation of linear operators between locally convex spaces. For this a natural bilinear integral for u-simple functions is used.

In [1] several theorems of representation of linear operators are proved using a bilinear integral in locally convex spaces similar to the integral used in [6] and [9]. Here we obtain several results of the same type using a natural integral for u-simple functions, which allows us to assume the measure to be only finitely additive and of bounded semivariation. In comparison with [1] results stated here present several advantages, between them it must be mentionned that now it is necessary to have no control family of the measure (the measures of this control family in [1] are assumed to be Radon measures in several cases), so conditions given here are more natural and simple.

The integral used here is a particular case of the bilinear integral studied in [7] (which is not in general comparable with the integral of [6]).

Let X and Y be two complete locally convex Hausdorff spaces and denote by P and Q two generating families of seminorms of the topologies of X and Y respectively and by $L(X,Y)$ the space of all continuous linear maps from X into Y endowed with the topology of the point convergence. Let us consider the space $\mathbf{C} = \mathbf{C}(K,X)$ of all continuous functions from a compact Hausdorff space K into X, endowed with the topology defined by the family of seminorms

$$p_K(f) = \max\{p(f(t)) : t \in K\} \quad \text{with} \quad p \in P.$$

We will denote by $\mathcal{B}$ the σ-algebra of Borel subsets of K.

Definition 1. Let $m : \mathcal{B} \to L(X,Y)$ be a finitely additive measure, then for a simple function $f = \sum_{i=1}^{n} x_i \chi_{B_i} : K \to X$ we define as usual

$$\int_B f dm = \sum_{i=1}^{n} m(B \cap B_i)x_i \in Y \quad (B \in \mathcal{B})^{(1)}$$

For $p \in P$ and $q \in Q$ the semivariation $m_{q,p} : \mathcal{B} \to \mathbf{R}$ will be defined by

$$m_{q,p}(B) = \sup q(\int_B f dm) \quad (B \in \mathcal{B})$$

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(1) Now on we will write $x_i m(B \cap B_i)$ to denote $m(B \cap B_i)x_i$.

where the supremum is taken over all simple functions $f : K \to X$ such that

$$p_B(f) = \sup_{t \in B} p(f(t)) \leq 1.$$

It is easy verified that

$$q\left(\int_B f dm\right) \leq p_B(f) m_{q,p}(B) \tag{1.1}$$

for all $B \in \mathcal{B}, f : K \to X$ simple, $p \in P$ and $q \in Q$ such that $m_{q,p}(K) < +\infty$.

Definition 2. A function $f : K \to X$ is said to be *u-simple* if there exists a net $\{f_i\}_{i \in I}$ of simple functions such that for every $\epsilon > 0$ and every $p \in P$ it is possible to find i_0 for which it is verified that $p_K(f - f_i) < \epsilon$ whenever $i > i_0$. These nets are called approximating nets of f. Let us denote by S the set of u-simple functions from K into X. If m is of bounded semivariation, that is, for every $q \in Q$ there exists $p \in P$ such that $m_{q,p}(K) < +\infty$, we define the integral of $f \in \mathbf{S}$ as usual:

$$\int_B f dm = \lim_i \int_B f_i dm \quad (B \in \mathcal{B})$$

being $\{f_i\}_{i \in I}$ an approximating net of f. It follows immediately from (1.1) that the operator $T_m : S \to Y$, defined by the integral

$$T_m(f) = \int_B f dm \quad (f \in \mathbf{S}),$$

is linear and continuous if S is endowed with the topology defined by the family of semi-norms $\{p_K : p \in P\}$ as described in the case of functions belonging to $\mathbf{C}(K, X)$ (2)

Theorem 3. *Let $T : \mathbf{C}(K, X) \to Y$ be a continuous linear operator, then the following are equivalent:*

3.1. *There exists a finitely additive measure of bounded semivariation $m : \mathcal{B} \to L(X, Y)$ such that $T = T_m$.*

3.2. *There is a continuous linear operators $\bar{T} : S \to Y$ which extends T.*

Proof. 3.1. implies evidently 3.2. so let us suppose that 3.2. is verified. Then for every $B \in \mathcal{B}$ and $x \in X$ it is defined in a natural way $m(B)x = \bar{T}(x\chi_B)$ $(\in Y)$, and it is easily proved that $m(B) \in L(X, Y)$ and that the set function constructed is finitely additive and it represents the operator T (this is a trivial consequence of the continuity of $\bar{T}$). Moreover m is of bounded semivariation because for every $q \in Q$ there exists $p \in P$ such that $q(\bar{T}(f)) \leq p_K(f)$ for all $f \in \mathbf{S}$, and therefore,

$$\begin{aligned} m_{q,p}(K) &= \sup\{q\left(\int_K f dm\right) : f \quad \text{simple}, \quad p_K(f) \leq 1\} \\ &= \sup\{q(\bar{T}(f)) : f \quad \text{simple}, \quad p_K(f) \leq 1\} \leq 1 < +\infty. \end{aligned}$$

(2) Note that $\mathbf{C}(K, X) \subset \mathbf{S}$.

Definition 4. Let $T : C \to Y$ be a continuous linear operator. For $p \in P$ and $q \in Q$ we define the (p,q)-semivariation of T by:

$$T_{q,p}(B) = \inf_{G \in \pi(B)} \sup\{q(T(f)) : f \in \mathbf{C},\ \text{supp } f \subset G, p_K(f) \leq 1\}$$

for all $B \in \mathcal{B}$, being $\pi(B)$ the family of the open subsets of K containing B.

Since T is continuous, for every $q \in Q$ there exists $p \in P$ such that $T_{q,p}(K) < +\infty$. Moreover it is easily proved that the set function $T_{q,p}$ is finitely subadditive and monotone on $\mathcal{B}$.

Let $\pi_0(B)$ the family of compact subsets of K contained in B $(\in \mathcal{B})$ and

$$\{G-H\}_B = \{G-H : G \in \pi(B), H \in \pi_0(B)\}.$$

Let $F : \mathcal{B} \to \mathbf{R}$ be a function. It will be said that

$$\lim_{\{G-H\}_B} F(G-H) = 0$$

if for every $\epsilon > 0$ there is $G_0 - H_0 \in \{G-H\}_B$ such that $F(G-H) < \epsilon$ holds for every $G - H \subset G_0 - H_0$.

We say that a continuous linear operator $T : \mathbf{C} \to Y$ is (G,H)-*continuous* if

$$\lim_{\{G-H\}_B} T_{q,p}(G-H) = 0 \tag{4.1}$$

holds for every $B \in \mathcal{B}, p \in P$ and $q \in Q$ with $T_{q,p}(K) < +\infty$. Last condition is equivalent to the following:

$$\sup_{H \in \pi_0(B)} T_{q,p}(H) = T_{q,p}(B)$$

for all $B \in \mathcal{B}, p \in P$ and $q \in Q$ with $T_{q,p}(K) < +\infty$.

Analogously, if $m : \mathcal{B} \to L(X,Y)$ is a finitely additive measure of bounded semivariation, we say that m is (G,H)-*continuous* if

$$\lim_{\{G-H\}_B} m_{q,p}(G-H) = 0 \tag{4.3}$$

holds for every $B \in \mathcal{B}, p \in P$ and $q \in Q$ such that $m_{q,p}(K) < +\infty$. Last condition is equivalent to:

$$\sup_{H \in \pi_0(B)} m_{q,p}(H) = m_{q,p}(B) = \inf_{G \in \pi(B)} m_{q,p}(G)$$

for all $B \in \mathcal{B}, p \in P$ and $q \in Q$ with $m_{q,p}(K) < +\infty$.

Proosition 5. *If $m : \mathcal{B} \to L(X,Y)$ is a (G,H)-continuous measure then*

$$m_{q,p}(B) = \inf_{G \in \pi(B)} \sup\{q(T_m(f)) : f \in \mathbf{C},\ \textit{supp } f \subset G, p_K(f) \leq 1\}$$

holds for every $B \in \mathcal{B}, p \in P$ and $q \in Q$ such that $m_{q,p}(K) < +\infty$.

Proof. Let it be $B \in \mathcal{B}$ and $f \in \mathbf{C}$ with supp $f \subset G \in \pi(B)$ and $p_K(f) \leq 1$, then there exists an approximating net $\{f_i\}_{i \in I}$ of f such that supp $f_i \subset G$ and $p_K(f_i) \leq 1$ for all $i \in I$. Therefore,

$$q(T_m(f)) = \lim_i q(\int_K f_i dm) = \lim_i q(\int_G f_i dm) \leq m_{q,p}(G)$$

and

$$\inf_{G \in \pi(B)} \sup\{q(T_m(f)) : f \in \mathbf{C}, \text{ supp } f \subset G, p_K(f) \leq 1\} \leq \inf_{G \in \pi(B)} m_{q,p}(G) = m_{q,p}(B),$$

since m is (G, H)-continuous. Let see now that the opposite inequality also holds. Let G be an open subset of K and $g : K \to X$ a simple function with $p_G(g) \leq 1$, then there exists a simple function $f = \sum_{i=1}^n x_i \chi_{E_i}$ such that $p(x_i) \leq 1$ for $i = 1, \cdots, n, \{E_i\}_{i=1}^n \subset \mathcal{B}$ is a partition of G and

$$\int_G f dm = \int_G g dm.$$

If for every $i = 1, \cdots, n$ we choose a compact set K_i and an open set G_i such that $K_i \subset E_i \subset G_i \subset G$, then $\{G_i'\}_{i=0}^n$ is an open cover of K, being $G_0' = K - \bigcup_{i=1}^n K_i$ and $G_i' = G_i - \bigcup_{\substack{j=1 \\ i \neq j}}^n K_j$ for $i = 1, \cdots, n$, and there exists a continuous partition of the unity $\{f_i\}_{i=0}^n$ associated to that cover such that $f_i|_{K_i} = 1$, supp $f_i \subset G_i$ $(i = 1, \cdots, n)$ and $\sum_{i=0}^n f_i = 1$. So for every choice α of the families $\{K_i^\alpha\}_{i=0}^n, \{G_i^\alpha\}_{i=0}^n$ we have continuous partition of the unity $\{f_i^\alpha\}_{i=0}^n$ and a continuous function

$$f^\alpha = \sum_{i=0}^n x_i f_i^\alpha$$

from K into X such that $p_K(f^\alpha) \leq 1$, supp $f^\alpha \subset G$ and $f^\alpha - f|_{\bigcup_{i=1}^n K_i} = 0$. Then,

$$\begin{aligned} q(\int_G g dm) = q(\int_G f dm) &\leq q(\int_G f^\alpha dm) + q(\int_G (f^\alpha - f) dm) \\ &\leq q(\int_G f^\alpha dm) + 2 \sum_{i=1}^n m_{q,p}(G_i^\alpha - K_i^\alpha) \end{aligned}$$

and since m is (G, H)-continuous, it results that

$$q(\int_G g dm) \leq \sup\{q(T_m(f)) : f \in \mathbf{C}, \text{ supp } f \subset G, p_K(f) \leq 1\}.$$

Therefore,

$$\begin{aligned} m_{p,q}(B) &= \inf_{G \in \pi(B)} m_{q,p}(G) \\ &\leq \inf_{G \in \pi(B)} \sup\{q(T_m(f)) : f \in \mathbf{C}, \text{ supp } f \subset G, p_K(f) \leq 1\}. \end{aligned}$$

Theorem 6. *Let $T : \mathbf{C} \to Y$ be a (G,H)-continuous operator, then there is a continuous operator $\bar{T} : \mathbf{S} \to Y$ which extends T.*

Proof. Let $f = \sum_{i=1}^n x_i \chi_{E_i}$ be a simple function. Given two families $\{H_i^\alpha\}_{i=0}^n$ and $\{G_i^\alpha\}_{i=0}^n$ of subsets of K, compact and open respectively, such that $H_i^\alpha \subset E_i \subset G_i^\alpha$ $(i = 1, \cdots, n)$, we have an open cover of $K, \{G_i'^\alpha\}_{i=0}^n$, with $G_i'^\alpha = G_i^\alpha - \bigcup_{\substack{j=i \\ i \neq j}}^n H_j^\alpha$. Let $\{f_i^\alpha\}_{i=1}^n$ be a continuous partition of the unity associated to that cover and let us consider

$$f_\alpha = \sum_{i=1}^n x_i f_i^\alpha \,^{(3)}$$

It is easy to see that $p_K(f_\alpha) = p_K(f)$, therefore

$$p_K(f_\alpha - f_{\alpha'}) \leq p_K(f_\alpha) + p_K(f_{\alpha'}) = 2p_K(f).$$

Also $f_\alpha(t) = f(t)$ for all $t \in \bigcup_{i=1}^n H_i^\alpha$ and if we consider that $\alpha \leq \alpha'$ if and only if $H_i^\alpha \subset H_i^{\alpha'}$ and $G_i^\alpha \supset G_i^{\alpha'}$ for all $i = 1, \cdots, n$, we have a net $\{f_\alpha\} \subset \mathbf{C}$ such that the net $\{T(f_\alpha)\}$ is convergent as we are going to prove now. In fact, if $p \in P$ and $q \in Q$ are such that $T_{q,p}(K) < +\infty$,

$$q(T(f_\alpha) - T(f_{\alpha'})) = q(T(f_\alpha - f_{\alpha'})),$$

and, as supp $(f_\alpha - f_{\alpha'}) \subset K - \bigcup_{i=1}^n H_i^\alpha \cap H_i^{\alpha'}$, by Definition 4, we have

$$q(T(f_\alpha - f_{\alpha'})) \leq p_K(f_\alpha - f_{\alpha'}) T_{p,q}(K - \bigcup_{i=1}^n H_i^\alpha \cap H_i^{\alpha'}),$$

then

$$\begin{aligned} q(T(f_\alpha) - T(f_{\alpha'})) &\leq 2p_K(f) T_{q,p}(K - \bigcup_{i=1}^n H_i^\alpha \cap H_i^{\alpha'}) \\ &\leq 2p_K(f) \sum_{i=1}^n T_{q,p}[(G_i^\alpha \cup G_i^{\alpha'}) - (H_i^\alpha \cap H_i^{\alpha'})] \end{aligned}$$

from where it follows that

$$\lim_{\alpha,\alpha'} q(T(f_\alpha) - T(f_{\alpha'})) = 0,$$

since T is (G,H)-continuous.

So let us define

$$\bar{T}_0(f) = \lim_\alpha T(f_\alpha). \tag{6.1}$$

From the (G,H)-continuity of T it is easily proved that the last limit (6.1) is well defined and that $\bar{T}_0$ is a continuous linear operator, and therefore, since simple functions are dense in $\mathbf{S}$, there exists one and only one continuous linear extension $\bar{T} : S \to Y$ of $\bar{T}_0$ defined by

(3) Remark that if there is an open set $\mathbf{C} \subset K$ such that $E_i \subset G$ for all $i = 1, \cdots, n$, then it can be obtained that supp $f \subset G$.

$\bar{T}(f) = \lim_\beta \bar{T}_0(f_\beta)$ for every $f \in \mathbf{S}$, being $\{f_\beta\}$ an approximating net of f. Let us prove now that $\bar{T}$ extends T. Given $f \in \mathbf{C}, \{f^\beta\}$ an approximating net of f and $\epsilon > 0$, for every $p \in P$ and $q \in Q$ such that $T_{q,p}(K) < +\infty$, there is β_0 such that $p_K(f - f^\beta) < \epsilon$ for all $\beta \geq \beta_0$. For every β let us choose the families $\{H_i^{\beta,\alpha}\}_{i=1}^{m_\beta}$ and $\{G_i^{\beta,\alpha}\}_{i=1}^{m_\beta}$ and consider the associated continuous functions f_α^β (following the last definition of $\bar{T}_0$) for which it holds

$$\begin{aligned} p_K(f - f_\alpha^\beta) &\leq p_K(f - f^\beta) + p_K(f^\beta - f_\alpha^\beta) < \epsilon + 2p_K(f^\beta) \\ &\leq \epsilon + 2(p_K(f) + p_K(f^\beta - f)) < 3\epsilon + 2p_K(f) \quad \text{for all} \quad \beta \geq \beta_0. \end{aligned}$$

For every β let us also define the open sets

$$G_{\beta,\alpha}^1 = K - \bigcup_{i=1}^{m_\beta} H_i^{\beta,\alpha} \quad \text{and} \quad G_{\beta,\alpha}^2 = \{t \in K : p(f(t) - f_\alpha^\beta(t)) < 2\epsilon\}.$$

As $f_\alpha^\beta(t) = f^\beta(t)$ for every $t \in K - \bigcup_{i=1}^{m_\beta} H_i^{\beta,\alpha}$ we have $\bigcup_{i=1}^{m_\beta} H_i^{\beta,\alpha} \subset G_{\beta,\alpha}^2$ and so $\{G_{\beta,\alpha}^1, G_{\beta,\alpha}^2\}$ is an open cover of K, and let $\{f_1^{\beta,\alpha}, f_2^{\beta,\alpha}\}$ be a continuous partition of the unity associated to that cover. Then

$$\begin{aligned} q(T(f) - T(f_\alpha^\beta)) &= q(T((f - f_\alpha^\beta)(f_1^{\beta,\alpha} + f_2^{\beta,\alpha}))) \\ &\leq q(T(f_1^{\beta,\alpha}(f - f_\alpha^\beta))) + q(T(f_2^{\beta,\alpha}(f - f_\alpha^\beta))) \\ &\leq p_K(f - f_\alpha^\beta)T_{q,p}(K - \bigcup_{i=1}^{m_\beta} H_i^{\beta,\alpha}) + p_{G_{\beta,\alpha}^2}(f - f_\alpha^\beta)T_{q,p}(K) \\ &\leq (3\epsilon + 2p_K(f))T_{q,p}(\bigcup_{i=1}^{m_\beta}(G_i^{\beta,\alpha} - H_i^{\beta,\alpha})) + 2\epsilon T_{q,p}(K) \\ &\leq (3\epsilon + 2p_K(f))\sum_{i=1}^{m_\beta} T_{q,p}(G_i^{\beta,\alpha} - H_i^{\beta,\alpha}) + 2\epsilon T_{q,p}(K). \end{aligned}$$

From where it results that

$$\lim_\beta q(T(f) - \bar{T}_0(f^\beta)) = 0,$$

since $\{f^\beta\}$ is uniformly convergent and T is (G, H)-continuous.

Theorem 7. *The following assertions are equivalent:*

7.1. *$T : \mathbf{C} \to Y$ is (G, H)-continuous.*

7.2. *There exists a (G, H)-continuous measure $m : \mathcal{B} \to L(X, Y)$ such that $T(f) = \int_K f dm$ for all $f \in \mathbf{C}$.*

Moreover, this measure m is unique and regular, and $m_{q,p}(B) = T_{q,p}(B)$ for all $B \in \mathcal{B}, p \in P$ and $q \in Q$ with $T_{q,p}(K) < +\infty$.

Proof. It is easily proved that 7.2 implies 7.1. Let us suppose that 7.1. is verified, then from Theorems 3 and 6 it follows the existence of a finitely additive measure $m : \mathcal{B} \to$

$L(X,Y)$ with bounded semivariation which represents T. Let us see that m is (G,H)-continuous. If G is an open subset of K and f is a simple function with $p_G(f) \leq 1$, then proceeding like in the proof of Theorem 6, it can be found a net $(f^\alpha) \subset \mathbf{C}$ such that $p_K(f^\alpha) \leq 1$, supp $f^\alpha \subset G$ and

$$q(\int_G f dm) = \lim_\alpha q(T(f^\alpha)) \leq T_{q,p}(G),$$

being $p \in P$ and $q \in Q$ with $m_{q,p}(K) < +\infty$. Therefore, $m_{q,p}(G) \leq T_{q,p}(G)$ for every open subset G, from where it follows immediately that

$$\lim_{\{G-H\}_B} m_{q,p}(G-H) \leq \lim_{\{G-H\}_B} T_{q,p}(G-H) = 0$$

holds for all $B \in \mathcal{B}$ and so m is (G,H)-continuous.

Let us prove now that m is unique and regular. Suppose that $m, m' : \mathcal{B} \to L(X,Y)$ are two measures verifying 7.2, then if $x \in X$ and $B \in \mathcal{B}$, for every compact H and every open set G such that $H \subset B \subset G$, let $f_{H,G} : K \to [0,1]$ be a continuous function such that $f_{H,G}|_H \equiv 1$ and $f_{H,G}|_{K-G} \equiv 0$. Then for every $p \in P$ and $q \in Q$ which verify $T_{q,p}(K) < +\infty$, we have that

$$\begin{aligned} q(m(B)x - m'(B)x) &\leq q(\int_{G-H} (f_{H,G} - \chi_B) x dm) + q(\int_{G-H} (f_{H,G} - \chi_B) x dm') \\ &\leq 2p(x)[m_{q,p}(G-H) + m'_{q,p}(G-H)], \end{aligned}$$

from where it follows that $m(B)x = m'(B)x$, since m and m' are (G,H)-continuous.

Moreover if $B \in \mathcal{B}, q \in Q, x \in X$ and $\epsilon > 0$, then since m is (G,H)-continuous there exists a compact H and an open subset G such that $H \subset B \subset G$ and $m_{q,p}(G-H) < \epsilon$, being $p \in P$ with $m_{q,p}(K) < +\infty$. Therefore, we have

$$q(m(E)x) \leq p(x) m_{q,p}(E) \leq p(x) m_{q,p}(G-H) \leq p(x)\epsilon$$

for all $E \in \mathcal{B}$ with $E \subset G - H$, and then m is regular.

The equality of the semivariations of m and T follows immediately from Proposition 5.

Corollary 8. *Every (G,H)-continuous operator $T : \mathbf{C} \to Y$ has a unique (G,H)-continuous extension $\bar{T} : \mathbf{S} \to Y$.*

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